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ON STABILITY ESTIMATES AND REGULARIZATION OF BACKWARD INTEGER AND FRACTIONAL ORDER PARABOLIC EQUATIONS

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INTRODUCTION

1. Rationale

Parabolic equations backward in time with the integer and fractional orders are used to describe many important physical phenomena. For example, geophysical and geological processes, materials science, hydrodynamics, image processing, describe transport by fluid flow in a porous environment.

In addition, the class of semilinear parabolic equations, $u_t + A(t)u(t) = f(t, u(t))$, also used to describe some important physical phenomena. For example: a) $f(t,u) = u\left(b - c\|u\|^2\right), c > 0$ in neurophysiological modeling of large nerve cell systems with action potential; b) $f(t,u) = -\sigma u/(1+au+bu^2)$, $\sigma, a, b > 0$, in enzyme kinetics; c) $f(t,u) = -|u|^p u, p \geqslant 1$ or $f(t,u) = -u^p$ in heat transfer processes; d) $f(t,u) = au - bu^3$ as the AllenCahn equation describing the process of phase separation in multicomponent alloy systems or the GinzburgLandau equation in superconductivity; e) $f(t,u) = \sigma u(u-\theta)(1-u)(0 < \theta < 1)$ in population genetics. Besides, the Bürgers type equations backward in time is also frequently encountered in the applications of data assimilation, nonlinear wave process, in the theory of nonlinear acoustics or explosive theory and in the optimal control.

The problems mentioned above are often *ill-posed problems* in the sense of Hadamard. For inverse and ill-posed problems, if the final data of the problem is replaced small swaps, then it will lead to a problem that has no solution or its solution is far from the exact solution.

Therefore, giving stability estimates, regularization method, as well as effective numerical methods for finding approximate solutions for ill-posed problems, are always topical issues. For the above reasons, we choose research topics for our thesis was: ***"On stability estimates and regularization of***

backward integer and fractional order parabolic equations".

2. Research purposes

Our goal is to establish new results about stability estimates and regularization for backward integer and fractional order parabolic equations.

3. Research subjects

For the parabolic equations of the integer order, we focus on research Bürgers type equations backward in time, semilinear parabolic equations backward in time. For the parabolic equations of the fractional order, we focus on research linear equations.

4. Research scopes

We study stability estimates and regularization for parabolic equations backward in time of the integer and fractional order.

5. Research Methods

We use the well-known methods such as *logarithmically convex method*, *non-local boundary value problem method*, *Tikhonov regularization method* and *mollification method.*

6. Scientific and practical meaning

The thesis has achieved some new results on stability estimates and regularization for nonlinear parabolic equations backward in time of the integer order and linear parabolic equations backward in time of the fractional order. Therefore, the thesis contributes to enriching the research results in the field of inverse and ill-posed problems.

The thesis can serve as a reference for students, graduated students and other interested persons in mathematics.

7. Overview and structure of the thesis

7.1. Overview of some issues related to the thesis

Inverse and ill-posed problems appeared from the 50s of the last century. The first mathematicians addressed this problem are Tikhonov A. N., Lavrent'ev M. M., John J., Pucci C., Ivanov V. K. Especially, in 1963, Tikhonov A. N. gave a regularization method under his name for inverse and ill-posed problems. Since then, inverse and ill-posed problems have become a separate discipline of physics and computational science.

Consider semilinear parabolic equations backward in time

$$\begin{cases} u_t + Au = f(t,u), \quad 0 < t \le T, \\ \|u(T) - \varphi\| \le \varepsilon \end{cases} \tag{1}$$

with noise level ε.

Note that, there were many results of stability estimates and regularization for the problem in case $f = 0$. For linear problems, some methods can be included to be the quasi-reversibility method, Sobolev equation method, regularization Tikhonov method, nonlocal boundary value problem method, mollification method. However, for nonlinear problems, there are still many issues that need to be studied. For example, looking for stability estimates and regularization for equations with time-dependent coefficients are still open.

In 1994, Nguyen Thanh Long and Alain Pham Ngoc Dinh examined the ill-posed problem for parabolic equations of semilinear form (1). By using the theory of contraction semigroups and the strongly continuous generator is defined by the operator

$$A_\beta = -A(I + \beta A)^{-1}, \beta > 0,$$

they achieved an error of the logarithm type in $(0,1]$ between the solution of the original problem and the solution of the regularized problem.

In 2009, Dang Duc Trong et al considered problem (1) in one-dimensional space

$$\begin{cases} u_t - u_{xx} = f(x,t,u(x,t)), \ (x,t) \in (0,\pi) \times (0,T), \\ u(0,t) = u(\pi,t) = 0, \ t \in (0,T), \\ \|u(x,T) - \varphi\| \le \varepsilon, \end{cases} \tag{2}$$

where f satisfies the global Lipschitz condition. These authors have use the integral equation method to regularize equation (2). Specifically, they regularized problem (2) by following problem

$$u^\epsilon(x,t) = \sum_{n=1}^{\infty}(\epsilon n^2 + e^{-Tn^2})^{\frac{t-T}{T}} \left(\varphi_n - \int_t^T e^{(s-T)n^2} f_n(u^\epsilon)ds\right) \sin nx. \tag{3}$$

with condition

$$\sum_{n=1}^{\infty} n^4 e^{2Tn^2} |\left\langle u(t), \phi_n \right\rangle|^2 < \infty, \ \forall t \in [0,T], \tag{4}$$

where $\phi_n = \sin(nx)$. These authors achieved an error of Hölder type that is as follows

$$\|u(t) - u^\epsilon(t)\| \leq M e^{k^2 T(T-t)} \epsilon^{\frac{t}{T}} \left(\frac{T}{1+\ln\frac{T}{\epsilon}}\right)^{1-t/T}.$$

In 2010, Phan Thanh Nam regularized for problem (1) by the spectral method. Author considered A as a positive self-adjoint unbounded linear operator and H is an orthonormal eigenbasis $\{\phi_i\}_{i\geqslant 1}$ corresponding to the eigenvalues $\{\lambda_i\}_{i\geqslant 1}$ such that

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots, \quad \text{and} \quad \lim_{i\to+\infty} \lambda_i = +\infty \tag{5}$$

and f satisfies the global Lipschitz condition. Phan Thanh Nam proved the following problem is well-poosed

$$\begin{cases} v_t + Av = P_M f(t, v(t)), & 0 < t < T, \\ v(T) = P_M g \end{cases} \tag{6}$$

where

$$P_M w = \sum_{\lambda_n \leq M} \left\langle \phi_n, w \right\rangle \phi_n$$

and achieved the following results:
If $\sum\limits_{n=1}^{\infty} e^{2\lambda_n \min(t,\beta)} |(u(t),\phi_n)|^2 \leqslant E_0^2$ then with $\beta \geq T$, we have

$$\|v(t) - u(t)\| \leq c\epsilon^{t/T}.$$

If $\sum\limits_{n=1}^{\infty} \lambda_n^{2\beta'} e^{2\lambda_n \min(t,\beta)} |(u(t),\phi_n)|^2 \leqslant E_1^2$ then with $\beta \geq T$ we have

$$\|v(t) - u(t)\| \leq c\epsilon^{t/T} \max\left\{\ln(1/\epsilon)^{-\beta'}, \epsilon^{(\tau - T)/\tau}\right\}.$$

If $\sum\limits_{n=1}^{\infty} e^{2\lambda_n}|(u(t),\phi_n)|^2 \leqslant E_2^2$ then

$$\|v(t)-u(t)\| \le c\epsilon^{t/T}\max\left\{\epsilon^{(\beta-T)/\tau},\epsilon^{(\tau-T)/\tau}\right\}.$$

In 2014, Nguyen Huy Tuan and Dang Duc Trong considered the problem (1) with A satisfies conditions like Phan Thanh Nam. For $v\in H$, they give a definition

$$A_\varepsilon(v)=\sum_{k=0}^{\infty}\ln^+\Big(\frac{1}{\varepsilon\lambda_k+e^{-\lambda_k}}\Big)\langle v,\phi_k\rangle\phi_k$$

where $\ln^+(x) = \max\{\ln x, 0\}$. Moreover, they assume that f satisfies the following conditions

(F0) There exists a constant $L_0 \geqslant 0$ such that

$$\langle f(t,w_1)-f(t,w_2),w_1-w_2\rangle + L_0\|w_1-w_2\|^2 \geqslant 0.$$

(F1) For $r>0$, there exists a constant $K(r)\geqslant 0$ such that $f:\mathbb{R}\times H\to H$ there exists a constant locally Lipschitz

$$\|f(t,w_1)-f(t,w_2)\| \leqslant K(r)\|w_1-w_2\|$$

with $w_1, w_2 \in H$ and $\|w_i\| \leqslant r, i=1,2$.

(F2) $f(t,0)=0$ for all $t\in[0,T]$.

Nguyen Huy Tuan and Dang Duc Trong regularized problem (1) by problem

$$\begin{cases}\dfrac{dv_\varepsilon(t)}{dt}+A_\varepsilon v_\varepsilon(t)=f(v_\varepsilon(t),t), \quad 0<t<T,\\ v_\varepsilon(T)=\varphi.\end{cases} \tag{7}$$

These authors needed conditions

$$E^2=\int_0^T\sum_{k=1}^{\infty}\lambda_k^2 e^{2\lambda_k}\big|\langle u(s),\phi_k\rangle\big|^2<\infty.$$

They proved that the convergence rate of the regularized solutions to exactly solution is the same as $\varepsilon^{t/T}\Big(\ln\frac{e}{\varepsilon}\Big)^{t/T-1}$.

In 2015, Dinh Nho Hao and Nguyen Van Duc regularized problem (1) by non-local boundary value problem

$$\begin{cases} v_t + Av = f(t, v(t)), & 0 < t < T, \\ \alpha v(0) + v(T) = \varphi, & 0 < \alpha < 1. \end{cases} \tag{8}$$

Dinh Nho Hao and Nguyen Van Duc considered f that satisfies the global Lipschitz condition

$$\|f(t, w_1) - f(t, w_2)\| \leqslant k\|w_1 - w_2\| \tag{9}$$

with Lipschitz constant$k \in [0, 1/T)$ independent on t, w_1, w_2.

Moreover, with the assumption $\|u(0)\| \leqslant E, E > \varepsilon$, Dinh Nho Hao and Nguyen Van Duc obtain

$$\|u(\cdot, t) - v(\cdot, t)\| \leqslant C\varepsilon^{t/T} E^{1-t/T}, \quad \forall t \in [0, T]. \tag{10}$$

Dinh Nho Hao and Nguyen Van Duc are the first authors to achieve form speed Hölder when regularized for problem (1) only on condition $\|u(0)\| \leq E$. However, this is true only Lipschitz constant $k \in [0, 1/T)$.

In addition to the semi-linear parabolic equation, Bürgers type equations backward in time is also of interest to many mathematicians. Abazari R., Borhanifar A., Srivastava V. K., Tamsir M., Bhardwaj U., Sanyasiraju Y., Zhanlav T., Chuluunbaatar O., Ulziibayar V., Zhu H., Shu H., Ding M. gave the numerical method for Bürgers equations. Allahverdi N. et al consider the application of Bürgers equation in optimal controlxt. Lundvall J. et al consider the application of Bürgers equation in assimilating data. Carasso A. S., Ponomarev S. M. use logarithmically convex method to give stability estimates for Bürgers equation.

Different from the parabolic equations backward in time of integer order, the parabolic equations backward in time of fractional order appear later, but they are also a very exciting research direction in recent years. Mathematicians have achieved a number of important results in the direction of this study. For example, Sakamoto K. and Yamamoto M. Have achieved results of the existence and unique inconsistency of the experiment, and their associates have achieved a stable evaluation result by the Carleman's evaluation method.

Regularization methods and efficient numerical methods for fractional parabolic equations backward in time was also proposed by mathematicians like non-local boundary value problem method, Tikhonov regularization method, spectral method, quasi-reversibility method, differential methods, finite element methods, variational methods, and some other methods.

7.2. Organization of the research

The main content of the thesis is presented in 4 chapters.

Chapter 1, we present the basic knowledge and some complementary knowledge, which are used in the following chapters.

Chapter 2, we state the obtained new results of stability estimates and Tikhonov regularization for backward integer order semilinear parabolic equations.

Chapter 3, we state the obtained new results of stability estimates for Bürgers-type equations backward in time.

Chapter 4, we state the obtained new regularization for fractional parabolic equations backward in time by mollification method.

The main results of the thesis were presented at the seminar of the Analysis Department , Institute of Natural Pedagogy - Vinh University, at the seminar of the differential equation Departement, Institute of Mathematics, Vietnam Academy of Science and Technology, and at Scientific workshop "Optimal and Scientific Calculation 15th" at Ba Vi from 20-22/4/2017. The results of the thesis were also reported at the 9th Vietnam Mathematical Congress in Nha Trang 14-18/8/2018.

These results have published in 04 articles, including 01 article on *Inverse Problems* (SCI), 01 article on *Journal of Inverse and Ill-Posed Problems* (SCIE), 02 article on *Acta Mathematica Vietnamica* (Scopus).

CHAPTER 1

BASIC KNOWLEDGE

1.1 Concepts of ill-posed problem, stability estimates and regularization

This section presents the concepts of ill-posed problem, stability estimates and regularization.

1.2 Auxiliary results

This section, outlines some of the knowledge needed for the following chapters.

Definition 1.2.3. The Gamma function Γ is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \tag{1.1}$$

whit z belongs to the right half plane $\mathrm{Re} z > 0$ of the complex plane.

Definition 1.2.5. The function $E_{\alpha,\beta}(z)$ is given by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, z \in \mathbb{C},$$

where $\alpha > 0, \beta > 0$ and Γ is Gamma function is called *Mittag-Leffler function.*

Definition 1.2.7. Cho f is differentiable continuous function on $[0, T]$ $(T > 0)$. Caputo fractional derivative with $\gamma \in (0, 1)$ of function f on $(0, T]$ is given by

$$\frac{d^\gamma}{dt^\gamma} f(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} \frac{d}{ds} f(s) ds, \ 0 < t \leqslant T.$$

Definition 1.2.11. The function $D_\nu(x) = \prod\limits_{j=1}^{n} \dfrac{sin(\nu x_j)}{x_j}$ $(\nu > 0)$ is called *Dirichlet kernel.*

CHAPTER 2

STABILITY ESTIMATES FOR SEMILINEAR PARABOLIC EQUATIONS BACKWARD IN TIME

In this chapter, we give stability estimates for semilinear parabolic equations backward in time. Then, we use the Tikhonov method to regularize this equation. Our results in this chapter are the first results on stability estimates, regularization for semilinear parabolic equations backward in time (Lipschitz constant nonnegative arbitrary) under only with a condition of the bounded solution at $t = 0$. These results were published in

- Duc N. V. , Thang N. V. (2017), Stability results for semi-linear parabolic equations backward in time, *Acta Mathematica Vietnamica* 42, 99-111.
- Ho D. N., Duc N. V. and Thang N. V. (2018), Backward semi-linear parabolic equations with time-dependent coefficients and locally Lipschitz source, *J. Inverse Problems* 34, 055010, 33 pp.

2.1 Stability estimates for semilinear parabolic equations backward in time with time-dependent coefficients

Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. We suppose that the operator $A(t)$ satisfies the following conditions:

(A1) $A(t)$ is a positive self-adjoint unbounded operator on H for each $t \in [0, T]$.

(A2) If $u_1(t), u_2(t)$ are two solutions of the equation

$$Lu = \frac{du}{dt} + A(t)u = f(t, u), \quad 0 < t \leq T, \tag{2.1}$$

then there exist a continuous function $a_1(t)$ on $[0,T]$ with $c \leqslant a_1(t) \leqslant c_1, \forall t \in [0,T]$, and a constant c_2 such that $w = u_1 - u_2$ satisfies the inequality

$$-\frac{d}{dt}\langle A(t)w, w\rangle \geqslant -2\langle A(t)w, w_t\rangle - a_1(t)\langle A(t)w, w\rangle - c_2\|w\|^2.$$

With $t \in [0,T]$, set

$$a_2(t) = \exp\left(\int_0^t a_1(\tau)d\tau\right), \quad a_3(t) = \int_0^t a_2(\xi)d\xi$$

and

$$\nu(t) = \frac{a_3(t)}{a_3(T)}. \tag{2.2}$$

First, stability estimates with the bound solution in $[0,T]$. Suppose f satisfies the condition (F1) as follows.

(F1) For each $r > 0$, there exists a constant $K(r) \geqslant 0$ such that $f : [0,T] \times H \to H$ satisfies the local Lipschitz condition

$$\|f(t,w_1) - f(t,w_2)\| \leqslant K(r)\|w_1 - w_2\|$$

for every $w_1, w_2 \in H$ such that $\|w_i\| \leqslant r, i = 1,2.$

Theorem 2.1.2. *Suppose that the operator $A(t)$ satisfies the conditions (A1),(A2) and the function f satisfies the condition (F1). Let u_1 and u_2 be two solutions of the problem* (2.1) *satisfying $\|u_i(T) - \varphi\| \leqslant \varepsilon$ and the constraint*

$$\|u_i(t)\| \leqslant E, \quad t \in [0,T], \quad i = 1,2, \quad 0 < \varepsilon < E. \tag{2.3}$$

Then for $t \in [0,T]$ we have

$$\|u_1(t) - u_2(t)\| \leqslant 2\varepsilon^{\nu(t)}E^{1-\nu(t)}\exp\Big(c_3\nu(t)(1-\nu(t))\Big), \tag{2.4}$$

where

$$c_3 = \left(\frac{1}{2}K^2T + |c_2|T + 2K\right)c_4c_5$$

with $c_4 = \frac{a_3(T)}{T}, c_5 = \max\{\exp|c_1|T, \exp|c|T\}$ and $K = K(E)$, the Lipschitz constant in (F1).

The stability estimate in Theorem 2.1.2 provides no information at $t=0$. For getting it, we require more conditions on $A(t)$ and stronger bounds for solutions. We have the following results.

Theorem 2.1.7. *Let A be a positive self-adjoint unbounded operator admitting an orthonormal eigenbasis $\{\phi_i\}_{i\geqslant 1}$ in H associated with the eigenvalues $\{\lambda_i\}_{i\geqslant 1}$ such that $0<\lambda_1<\lambda_2<\dots$ and $\lim\limits_{i\to+\infty}\lambda_i=+\infty$. Let $a(t)$ be a continuously differentiable function in $[0,T]$ such that $0<a_0\leqslant a(t)\leqslant a_1$ and $M=\max\limits_{t\in[0,T]}|a_t(t)|<+\infty$. Suppose that f satisfies the condition (F1), u_1 and u_2 are two solutions of the problem $u_t+a(t)Au=f(t,u(t)), 0<t\leqslant T$ such that $\|u_i(T)-\varphi\|\leqslant\varepsilon,\quad i=1,2$. Then the following stability estimates hold:*
i) If

$$\sum_{n=1}^{\infty}\lambda_n^{2\beta}\left\langle u_i(t),\phi_n\right\rangle^2\leqslant\overline{E}^2, t\in[0,T], i=1,2, \tag{2.5}$$

with $\overline{E}>\varepsilon$ and $\beta>0$ then

$$\|u_1(t)-u_2(t)\|\leq C_1(t)\varepsilon^{\nu(t)}\overline{E}^{1-\nu(t)}\left(\left(\ln\frac{\overline{E}}{\varepsilon}\right)^{-\beta}+\sqrt{\frac{\varepsilon}{\overline{E}}}\right)^{1-\nu(t)},\ t\in[0,T],$$

where $\nu(t)=\dfrac{\int_0^t a(\xi)d\xi}{\int_0^T a(\xi)d\xi}$ and $C_1(t)$ is a bounded function in $[0,T]$.

ii) If

$$\sum_{n=1}^{\infty}e^{2\gamma\lambda_n}\left\langle u_i(t),\phi_n\right\rangle^2\leqslant\widetilde{E}^2, t\in[0,T], i=1,2 \tag{2.6}$$

with $\widetilde{E}>\varepsilon$ and $\gamma>0$ then

$$\|u_1(t)-u_2(t)\|\leqslant C_2(t)\varepsilon^{\nu_1(t)}\widetilde{E}^{1-\nu_1(t)},\ t\in[0,T],$$

where $\nu_1(t)=\dfrac{\gamma+\int_0^t a(\xi)d\xi}{\gamma+\int_0^T a(\xi)d\xi}$ and $C_2(t)$ is a bounded function in $[0,T]$.

In Theorem 2.1.7, we require the bound solution in $[0,T]$. It is better to change them by those at $t=0$. For this purpose, we assume:

(F2) $f(t,0)=0$ with forall $t\in[0,T]$.

(F3) There exists a constant $L_1\geqslant 0$ such that

$$\left\langle f(t,w_1)-f(t,w_2),w_1-w_2\right\rangle\leqslant L_1\|w_1-w_2\|^2.$$

Theorem 2.1.11. *Suppose that the operator $A(t)$ satisfies the conditions (A1),(A2) and f satisfies the conditions (F1)–(F3). Let u_1 and u_2 be two solutions of the problem* (2.1) *satisfying the constraints $\|u_i(T)-\varphi\| \leqslant \varepsilon$ and*

$$\|u_i(0)\| \leqslant E, \quad i=1,2,$$

with $0<\varepsilon<E$, then

$$\|u_1(t)-u_2(t)\| \leqslant 2\exp\Big(\Big(\frac{1}{2}K^2T+|c_2|T+2K\Big)c_4c_5\nu(t)(1-\nu(t))\Big) \\ \times \varepsilon^{\nu(t)}E^{1-\nu(t)}, \forall t\in[0,T]$$

where $c_4=\frac{a_3(T)}{T}$, $c_5=\max\{\exp|c_1|T, \exp|c|T\}$ v $K=K(e^{L_1T}E)$ the Lipschitz constant in (F1).

In the previous sections, we do not assume any relationship between the operator $A(t)$ and the function f. To enlarge the class of source functions f and to obtain stronger results, instead of (F1) we now assume:

(F4) For each $r>0$ and any solutions u_1 and u_2 of the problem (2.1) with $\langle A(t)u_i, u_i\rangle \leqslant r^2, i=1,2$ $t\in[0,T]$, there exists a constant $K(r)\geqslant 0$ such that $f:\mathbb{R}\times H\to H$ satisfies the condition

$$\|f(t,u_1)-f(t,u_2)\| \leqslant K(r)\|u_1-u_2\|.$$

(F5) There exists a constant $L_2 \geqslant 0$ such that, for any solution u of the problem (2.1),

$$\langle A(t)u, f(t,u)\rangle \leqslant L_2\langle A(t)u,u\rangle.$$

We have the following results

Theorem 2.1.14. *Suppose that the conditions (A1),(A2), (F2)–(F5) are satisfied and there exists a constant $L_3>0$ such that*

$$\langle A(0)u(0), u(0)\rangle \geqslant L_3\|u(0)\|^2.$$

If u_1 and u_2 are two solutions of the problem (2.1) satisfying the constraints $\|u_i(T)-\varphi\| \leqslant \varepsilon$ and

$$\langle A(0)u_i(0), u_i(0)\rangle \leqslant E_1^2, \quad i=1,2 \tag{2.7}$$

with $0 < \varepsilon < E_1$, *then for* $t \in [0,T]$ *there exists a bounded function* $\widetilde{C}(t)$ *such that*

$$\|u_1(t) - u_2(t)\| \leqslant \widetilde{C}(t)\varepsilon^{\nu(t)} E_1^{1-\nu(t)}. \tag{2.8}$$

Theorem 2.1.15. *Let operator* A *and function* $a(t)$ *satisfied conditions as in Theorem 2.1.7. Suppose that* f *satisfies the condition (F2)–(F5),* u_1 *and* u_2 *are two solutions of the problem* $u_t + a(t)Au = f(t,u(t)), 0 < t \leqslant T$ *such that* $\|u_i(T) - \varphi\| \leqslant \varepsilon, \quad i = 1,2$. *Then the following stability estimates hold:*

i) If

$$\sum_{n=1}^{\infty} \lambda_n^{2\beta} \left\langle u_i(0), \phi_n \right\rangle^2 \leqslant \overline{E}^2, i = 1,2 \tag{2.9}$$

with $\overline{E} > \varepsilon$ *and* $\beta \geqslant \dfrac{1}{2}$, *then there exists a bounded function* $\overline{C}(t)$ *in* $[0,T]$ *such that*

$$\|u_1(t) - u_2(t)\| \leqslant \overline{C}(t)\varepsilon^{\nu(t)}\overline{E}^{1-\nu(t)} \left(\left(\ln \frac{\overline{E}}{\varepsilon} \right)^{-\beta} + \sqrt{\frac{\varepsilon}{\overline{\overline{E}}}} \right)^{1-\nu(t)}, \tag{2.10}$$

where $\nu(t) = \dfrac{\int_0^t a(\xi) d\xi}{\int_0^T a(\xi) d\xi}$.

ii) If

$$\sum_{n=1}^{\infty} e^{2\gamma\lambda_n} \left\langle u_i(0), \phi_n \right\rangle^2 \leqslant \widetilde{E}^2, i = 1,2 \tag{2.11}$$

with $\widetilde{E} > \varepsilon$ *and* $\gamma > 0$, *then there exists a bounded defined function* $\overline{C}_1(t)$ *in* $[0,T]$ *such that*

$$\|u_1(t) - u_2(t)\| \leqslant \overline{C}_1(t)\varepsilon^{\nu_1(t)}\widetilde{E}^{1-\nu_1(t)}, \tag{2.12}$$

where $\nu_1(t) = \dfrac{\gamma + \int_0^t a(\xi) d\xi}{\gamma + \int_0^T a(\xi) d\xi}$.

2.2 Examples

In this section, we present some examples to illustrate assumptions we set in section 2.1. These examples also indicate that the theorem of stability estimates in section 2.1 is an application for some important physics problems such as in neurophysiological modeling of large nerve cell systems with action potential, in heat transfer processes, in population genetics, Ginzburg-Landau problem, in enzyme kinetics.

2.3 Stability estimates for semilinear parabolic equations backward in time with time-independent coefficients

In section 1.1, we have given stability estimates for semilinear parabolic equations backward in time with time-dependent coefficients and source function locally Lipschitz. These results lead to stability estimates for semilinear parabolic equations backward in time with time-dependent coefficients and source function global Lipschitz. However, in Theorem 2.1.2 and Theorem 2.1.7, in order to give stability estimates then we need condition of the bounded solution on domain $[0, T]$. In Theorem 2.1.11, Theorem 2.1.14 and Theorem 2.1.15, in order to give stability estimates only with the condition of the bounded solution at $t = 0$ then we need condition f satisfied (F2), i.e. $f(t, 0) = 0$. Therefore, the purpose of this section is to give stability estimates for semilinear parabolic equations backward in time with time-independent coefficients and source function satisfied condition Lipschitz

$$\|f(t, w_1) - f(t, w_2)\| \leq k\|w_1 - w_2\|, \quad w_1, w_2 \in H, \tag{2.13}$$

for some non-negative constant k independent of t, w_1 and w_2, only with condition of bounded solution at $t = 0$.

Let A be a positive self-adjoint unbounded linear operator on domain $D(A) \subset H$. Consider semilinear parabolic equations backward in time

$$\begin{cases} u_t + Au = f(t, u), \quad 0 < t \leq T, \\ \|u(T) - \varphi\| \leq \varepsilon \end{cases} \tag{2.14}$$

where φ is the final data of the problem determined by measurement of noise level ε and solutiion $u \in C^1((0, T), H) \cap C([0, T], H)$.

Now, we present the results of stability estimates.

Theorem 2.3.1. *Suppose u_1 and u_2 be two solutions of the problem* (2.14) *and f satisfies the condition* (2.13)*. If $u_i(0) \in D(A)$, $i = 1, 2$, and*

$$\|u_i(0)\| \le E, \quad i = 1, 2, \tag{2.15}$$

with $E > \varepsilon$, then with $t \in [0, T]$ have

$$\|u_1(t) - u_2(t)\| \le 2\varepsilon^{t/T} E^{1-t/T} \exp\left[\left(2k + \frac{1}{4}k^2(T+t)\right)\frac{t(T-t)}{T}\right]. \tag{2.16}$$

Theorem 2.3.3. *Assume that A admits an orthonormal eigenbasis $\{\phi_i\}_{i\geqslant 1}$ in H associated with the eigenvalues $\{\lambda_i\}_{i\geqslant 1}$ such that $0 < \lambda_1 < \lambda_2 < \dots$ and $\lim\limits_{i\to+\infty} \lambda_i = +\infty$. Suppose that f satisfies the Lipschitz condition* (2.13)*, u_1 and u_2 are solutions of the problem* (2.14) *with $u_i(0) \in D(A)$, $i = 1, 2$.*

i) If

$$\sum_{n=1}^{\infty} \lambda_n^{2\beta} \langle u_i(0), \phi_n\rangle^2 \leqslant E_1^2, i = 1, 2, \ \beta > 0 \tag{2.17}$$

with $E_1 > \varepsilon$ then with forall $t \in [0, T]$, there exists a bounded defined function $C(t)$ such that

$$\|u_1(t) - u_2(t)\| \le C(t)\varepsilon^{t/T} E_1^{1-t/T} \left(\left(\ln\frac{E_1}{\varepsilon}\right)^{-\beta} + \sqrt{\frac{\varepsilon}{E_1}}\right)^{1-t/T}. \tag{2.18}$$

ii) If

$$\sum_{n=1}^{\infty} e^{2\gamma\lambda_n} \langle u_i(0), \phi_n\rangle^2 \leqslant E_2^2, i = 1, 2, \ \gamma > 0 \tag{2.19}$$

with $E_2 > \varepsilon$ then with forall $t \in [0, T]$, there exists a bounded defined function $C_1(t)$ such that

$$\|u_1(t) - u_2(t)\| \le C_1(t)\varepsilon^{\frac{\gamma+t}{\gamma+T}} E_2^{1-\frac{\gamma+t}{\gamma+T}}. \tag{2.20}$$

2.4 Regularization for semilinear parabolic equations backward in time by method Tikhonov

In this section, besides the assumptions (A1),(A2), we assume that $A(t)$ is a positive self-adjoint unbounded operator for each $t \in [0, T]$ and $-A(t)$ is a generator of a contraction semigroup and that $(A(t)+I))^{-1}$ is strongly continuously differentiable. Furthermore, $-A(t)$ is generator a unique evolution system $U(t,s), 0 \leqslant s \leqslant t \leqslant T$ which is a family of bounded linear operators from H into itself defined for $0 \leqslant s \leqslant t \leqslant T$ and strongly continuous in the two variables jointly.

We stabilize the backward problem

$$\begin{cases} u_t + A(t)u = f(t,u), \quad 0 \leqslant t \leqslant T, \\ \|u(T) - \varphi\| \leqslant \varepsilon \end{cases} \tag{2.21}$$

by a modified version of Tikhonov regularization.

Denote by $v(t)$ the solution of the initial problem

$$v_t + A(t)v = f(t,v), \quad 0 < t \leqslant T, \; v(0) = g \in D(A(t)). \tag{2.22}$$

To emphasize the dependence of the solution v on the initial data g sometime we write $v(t,g)$ instead of $v(t)$. If the condition $\|u(0)\| \leqslant E$ is satisfied and f is demi-continuous and maps bounded sets into bounded sets and satisfies the conditions (F1)–(F3), it is normally processed by minimizing the Tikhonov functional

$$J_\alpha(g) = \|v(T,g) - \varphi\|^2 + \alpha\|g\|^2 \tag{2.23}$$

with $g \in D(A(t))$ and α being the regularization parameter. However, as in many other nonlinear ill-posed problems, it is not clear to us if such a minimization problem admits a solution. We therefore modify this approach by solving an approximate minimization problem. Namely, set

$$I = \inf_{g \in D(A(t))} J_\alpha(g), \tag{2.24}$$

and for fixed $\tau > 0$ choose $\overline{g} \in D(A(t))$ such that

$$J_\alpha(\overline{g}) \leqslant I + \tau\varepsilon^2. \tag{2.25}$$

Further, if the condition $\langle A(0)u(0), u(0)\rangle \leqslant E_1^2$ is satisfied and f satisfies the conditions (F2)–(F5), then as above we take the Tikhonov functional

$$J_\beta(g) = \|v(T,g) - \varphi\|^2 + \beta \langle A(0)g, g\rangle, \quad \beta > 0, \tag{2.26}$$

where β being the regularization parameter. Set

$$I_1 = \inf_{g \in D(A(t))} J_\beta(g). \tag{2.27}$$

With for fixed $\tau > 0$, choose $\widetilde{g} \in D(A(t))$ such that

$$J_\beta(\widetilde{g}) \leqslant I_1 + \tau\varepsilon^2, \tag{2.28}$$

then the problem (2.28) always admits a solution.

Theorem 2.4.2. Suppose that f is demi-continuous and maps bounded sets into bounded sets and satisfies the conditions (F1)–(F3). If the problem (2.21) has a solution $u(t)$ with $u(0) \in D(A(t))$ satisfying

$$\|u(0)\| \leqslant E$$

and $v(t, \overline{g})$ is a solution of the problem (2.22) vi $g = \overline{g}$, then with $\alpha = \left(\frac{\varepsilon}{E}\right)^2$ there exists a positive constant C such that

$$\|u(t) - v(t, \overline{g})\| \leqslant C\varepsilon^{\nu(t)} E^{1-\nu(t)}, \quad t \in [0, T].$$

Theorem 2.4.3. Suppose that f is demi-continuous and maps bounded sets into bounded sets and satisfies the conditions (F2)–(F5) and $\langle A(0)u(0), u(0)\rangle \geqslant L_3\|u(0)\|^2$ with $u(t)$ being a solution of problem $u_t + A(t)u = f(t, u), 0 < t \leqslant T$. If the problem (2.21) has a solution $u(t)$ with $u(0) \in D(A(t))$ satisfying

$$\langle A(0)u(0), u(0)\rangle \leqslant E_1^2,$$

and $v(t, \widetilde{g})$ is a solution of the problem (2.22) with $g = \widetilde{g}$, then with $\beta = \left(\frac{\varepsilon}{E_1}\right)^2$ there exists a positive constant C_1 such that

$$\|u(t) - v(t, \widetilde{g})\| \leq C_1\varepsilon^{\nu(t)} E_1^{1-\nu(t)}, \quad t \in [0, T].$$

2.5 Conclusions of Chapter

In Chapter 2, we obtained the following main results:

- Given stability estimates for semilinear parabolic equations backward in time with time-dependent coefficients and different conditions of source functions and different constraints of the solution. Give examples to illustrate for hypotheses of operator $A(t)$ and source function locally Lipschitz f.
- Given stability estimates for semilinear parabolic equations backward in time with time-independent coefficients.
- Regularization for semilinear parabolic equations backward in time with time-dependent by method Tikhonov.

CHAPTER 3

STABILITY ESTIMATES FOR BÜRGERS-TYPE EQUATIONS BACKWARD IN TIME

In this chapter, we give stability estimates for Bürgers-type equations with type Hölder. These results are generalization and improvement of results Carasso and Ponomarev. Specifically, we give stability estimates for more general equations under weaker conditions than those conditions set by the aforementioned authors. These results were published in
Ho D. N., Duc N. V. and Thang N. V.(2015), Stability estimates for Burgers-type equations backward in time, *J. Inverse and Ill-Posed Problems* 23, 41-49.

Let $T > 0$. Set

$$D := \{(x,t) : \ 0 < x < 1, \ 0 < t < T\}$$

and $\overline{D}$ is closure of D.
In this chapter, for simplicity, we write $\|\cdot\|$ instead $\|\cdot\|_{L^2(0,1)}$.

3.1 Stability estimates for Bürgers-type equations backward in tim with time-dependent coefficients.

In this section, we give stability estimates for Bürgers-type equations backward in time with time-dependent coefficients

$$u_t = (a(x,t)u_x)_x - d(x,t)uu_x + f(x,t), \quad (x,t) \in D, \tag{3.1}$$

$$u(0,t) = g_0(t), \quad u(1,t) = g_1(t), \quad 0 \leqslant t \leqslant T, \tag{3.2}$$

where $a(x,t)$, $d(x,t)$, $g_0(t)$, $g_1(t)$, $f(x,t)$ are smooth functions, $a(x,t) \geqslant \overline{a} > 0, (x,t) \in \overline{D}$, $a_t(x,t)$, $d(x,t)$ v $d_x(x,t)$ are bounded on $\overline{D}$.
Theorem 3.1.1. *Suppose $u_1(x,t)$ and $u_2(x,t)$ be two solutions of the problem*

(3.1),(3.2) *satisfies*

$$\max_{(x,t)\in\overline{D}}\{|u_i|,|u_{ix}|\}\leqslant E,\ \ i=1,2. \tag{3.3}$$

Set

$$m=\max_{(x,t)\in\overline{D}}\frac{a_t(x,t)+2(dE)^2}{a(x,t)}$$

and

$$\mu(t)=\frac{t}{T}\ \textit{nu}\ m=0,\quad \mu(t)=\frac{e^{mt}-1}{e^{mT}-1}\ \textit{nu}\ m\neq 0. \tag{3.4}$$

If $\|u_1(\cdot,T)-u_2(\cdot,T)\|\leqslant\delta$, *there exists a bounded defined function* $k_1(t)$ *such that*

$$\|u_1(\cdot,t)-u_2(\cdot,t)\|\leqslant k_1(t)\delta^{\mu(t)}E^{1-\mu(t)},\ \forall t\in[0,T]. \tag{3.5}$$

3.2 Stability estimates for Bürgers-type equations backward in tim with time-independent coefficients.

In this section, we give stability estimates for Bürgers-type equations backward in time with time-independent coefficients.

Theorem 3.2.1. *Let* $u_1(x,t)$ *and* $u_2(x,t)$ *be smooth solutions of*

$$\begin{aligned} u_t&=\nu u_{xx}-\alpha uu_x+f(x,t),\quad (x,t)\in D,\\ u(0,t)&=g_0(t),\quad u(1,t)=g_1(t),\quad 0\leqslant t\leqslant T,\end{aligned}$$

where $\nu>0$, $\alpha\in\mathbb{R}$, *and* g_0,g_1,f *are smooth functions. If* $u_1,\ u_2$ *satisfy*

$$\max_{(x,t)\in\overline{D}}\{|u_i|,|u_{ix}|,|u_{it}|\}\leqslant E,\ \ i=1,2 \tag{3.6}$$

and $\|u_1(\cdot,T)-u_2(\cdot,T)\|_{L^2}\leqslant\delta$, *then exists a bounded defined function* $k_2(t)$ *such that*

$$\|u_1(\cdot,t)-u_2(\cdot,t)\|\leqslant k_2(t)\delta^{\frac{t}{T}}E^{1-\frac{t}{T}},\quad t\in[0,T]. \tag{3.7}$$

3.3 Conclusions of Chapter 3

In Chapter 3, we obtained the following main results:
- Stability estimates type Hölder for Burgers-type equations backward in tim with time-dependent coefficients.
- Stability estimates type Hölder for Burgers-type equations backward in tim with time-independent coefficients.

CHAPTER 4

REGULARIZATION FOR FRACTIONAL PARABOLIC EQUATION BACKWARD IN TIME

We study fractional backward heat equation $\mathbb{R}^n$

$$\begin{cases} \dfrac{\partial^\gamma u}{\partial t^\gamma} = \Delta u, \ x \in \mathbb{R}^n, t \in (0,T) \\ u(x,T) = \varphi(x), \ x \in \mathbb{R}^n \end{cases} \tag{4.1}$$

where $0 < \gamma < 1$, φ is unknown exact data and only noisy data φ^ε with

$$\|\varphi^\varepsilon(\cdot) - \varphi(\cdot)\|_{L_2(\mathbb{R}^n)} \leqslant \varepsilon \tag{4.2}$$

is available.

In this chapter, we study problem (4.1)-(4.2) in the general space $\mathbb{R}^n$ and regularize the problem by the mollification method

$$\begin{cases} \dfrac{\partial^\gamma v^\nu}{\partial t^\gamma} = \Delta v^\nu, \ x \in \mathbb{R}^n, t \in (0,T) \\ v^\nu(x,T) = S_\nu(\varphi^\varepsilon(x)), \ x \in \mathbb{R}^n, \end{cases} \tag{4.3}$$

where $\nu > 0$ and $S_\nu(\varphi^\varepsilon(x))$ is the convolution of $\varphi^\varepsilon(x)$ with Dirichlet kernel.

These results were published in:
Duc N. V., Muoi P. Q., Thang N. V., A molification method backward time-fractional heat equation, *Acta Math. Vietnam.* (Accepted)

4.1 Well-posed of regularization problem

In this section, we prove that problem (4.3) is well-posed.

Theorem 4.1.3. *With $\varphi^\varepsilon \in L^2(\mathbb{R}^n)$, the problem (4.3) has a unique solution $v^\nu \in L^2(\mathbb{R}^n)$ and there exists a constant C_3 such that*

$$\|v^\nu(\cdot,t)\| \leq C_3(1+\nu^2)\|\varphi^\varepsilon\|, \ t \in [0,T].$$

4.2 Convergence rates

In this section, It is well-known that the convergence rates of a regularization method are obtained under some smoothness conditions of the exact solution together with a rule of regularization parameter choice.

Theorem 4.2.3. *If $u(x,t)$ is solution of (4.1) satisfies*

$$\|u(\cdot,0)\|_{H^s(\mathbb{R})} \le E \tag{4.4}$$

then with $\nu = \left(\dfrac{E}{\varepsilon}\right)^{\frac{1}{s+2}}$, *there exists a constant* $\overline{C}_1 > 0$ *such that*

$$\|v^\nu(\cdot,t) - u(\cdot,t)\|_{H^l(\mathbb{R})} \leqslant \overline{C}_1 \varepsilon^{\frac{s-l}{s+2}} E^{\frac{l+2}{s+2}},\ 0 \le l < s,\ t \in [0,T]. \tag{4.5}$$

Theorem 4.2.5. *Suppose that* $0 < \varepsilon < \|\varphi^\varepsilon(\cdot)\|$. *Choose* $\tau > 1$ *such that* $0 < \tau\varepsilon < \|\varphi^\varepsilon\|$. *Then there exists a number* $\nu_\varepsilon > 0$ *such that*

$$\|v^{\nu_\varepsilon}(\cdot,T) - \varphi^\varepsilon(\cdot)\| = \tau\varepsilon. \tag{4.6}$$

Further, if the solution $u(x,t)$ of (4.1) satisfies (4.4) *then there exists a constant* $\overline{C}_2 > 0$ *such that*

$$\|v^{\nu_\varepsilon}(\cdot,t) - u(\cdot,t)\|_{H^l(\mathbb{R})} \leqslant \overline{C}_2 \varepsilon^{\frac{s-l}{s+2}} E^{\frac{l+2}{s+2}},\ 0 \le l < s,\ t \in [0,T]. \tag{4.7}$$

4.3 Example numerical

This section is devoted to illustrating the performance of our regularization method. These numerical examples are done on computers LENOVO, Microsoft Windows 10 Home with version MATLAB 2015a.

4.4 Conclusions of Chapter 4

In chapter 4, we obtained the following main results:
- Prove that regularization problem is well-posed.
- Give convergence type Hölder of the regularized solutions to the exact solution.
- Give examples number that illustrates the theory part.

GENERAL CONCLUSIONS AND RECOMMENDATIONS

General conclusions

The dissertation studies stability estimates and regularization for parabolic equations of the order integer and order fractional backward in time. Main results of the thesis are:

1. We state results of stability estimates for semilinear parabolic equations of the order integer backward in time (with Lipschitz constant $k \geq 0$ arbitrary). This is the first result required only bounded solutions at $t = 0$.

2. We state results of stability estimates and Tikhonov regularization for semilinear parabolic equations of the order integer with time-dependent coefficients backward in time and locally Lipschitz source.

3. Generalize and improve the results of Carasso and Ponomarev about stability estimates for type Bürgers equations.

4. Regularized in both a priori and a posteriori parameter choice rules for fractional parabolic equations backward in time by mollification method. After that, we give a numerical example to illustrate our theory.

Recommendations

In the future, we look forward to continuing to study the following issues:

1. Research about stability estimates and regularization for nonlinear parabolic equations of the order integer in Banach space.

2. Research about stability estimates and regularization for linear parabolic equations of the order fractional in Banach space and nonlinear parabolic equations of the order fractional in Hilbert space.

3. Research about the problem of determining the inverse source for parabolic equations.

LIST OF PUBLICATIONS RELATED TO THE THESIS

1. Ho D. N., Duc N. V. and Thang N. V.(2015) Stability estimates for Burgers-type equations backward in time, *J. Inverse and Ill-Posed Problems*, **23**, 41-49.

2. Duc N. V. and Thang N. V.(2017), Stability results for semi-linear parabolic equations backward in time, *Acta Math. Vietnam.*, **42**, 99–111.

3. Ho D. N., Duc N. V. and Thang N. V. (2018), Backward semi-linear parabolic equations with time-dependent coefficients and locally Lipschitz source, *Inverse Problems,* **34**, 055010, 33 pp.

4. Duc N. V. , Muoi P. Q. and Thang N. V., A mollification method for backward time-fractional heat equation, *Acta Math. Vietnam.* (Accepted)