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COTINUITY OF SOLUTION MAPPINGS FOR EQUILIBRIUM PROBLEMS

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PREFACE

1 Rationale

1.1. Stability of solutions for optimization related problems, including semicontinuity, continuity, Hölder/Lipschitz continuity and differentiability properties of the solution mappings to equilibrium and related problems is an important topic in optimization theory and applications. In recent decades, there have been many works dealing with stability conditions for optimization-related problems as optimization problems, vector variational inequality problems, vector quasiequilibrium problems, variational relation problems. In fact, differentiability of the solution mappings is a rather high level of regularity and is somehow close to the Lipschitz continuous property (due to the Rademacher theorem). However, to have a certain property of the solution mapping, usually the problem data needs to possess the same level of the corresponding property, and this assumption about the data is often not satisfied in practice. In addition, in a number of practical situations such as mathematical models for competitive economies, the semicontinuity of the solution mapping is enough for the efficient use of the models. Hence, the study of the semicontinuity and continuity properties of solution mappings in the sense of Berge and Hausdorff is among the most interesting and important topic in the stability of equilibrium problems.

1.2. The Painlevé-Kuratowski convergence plays an important role in the stability of solution sets when problems are perturbed by sequences constrained set and objective mapping converging. Since the perturbed problems with sequences of set and mapping converging are different from such parametric problems with the parameter perturbed in a space of parameters, the study of Painlevé-Kuratowski convergence of the solution sets is useful and deserving. Moreover, this topic is closely related to other important ones, including solution method, approximation theory. Therefore, there are many works devoted to the Painlevé-Kuratowski convergence of solution sets for problems related to optimization. Hence, the researching of convergence of solution sets in the sense of the Painlevé-Kuratowski is an important and interesting topic in optimization theory and applications.

1.3. Well-posedness plays an important role in stability analysis and numerical method in optimization theory and applications. In recent years, there have been many works dealing with stability conditions for optimization-related problems as optimization problems, vector variational inequality problems, vector quasiequilibrium problems. Recently, Khanh et al. (in 2014) introduced two types of Levitin-Polyak well-posedness for weak bilevel vector equilibrium and optimization problems with equilibrium constraints. Using the generalized level closedness conditions, the authors studied the Levitin-Polyak well-posedness for such problems. However, to the best of our knowledge, the Levitin-Polyak well-posedness and Levitin-Polyak well-posedness in the generalized sense for bilevel equilibrium problems and traffic network problems with equilibrium constraints are open problems. Motivated and inspired by the above observations, we have chosen the topic for the thesis that is: ***"Cotinuity of solution mappings for equilibrium problems***"

2 Subject of the research

The objective of the thesis is to establish the continuity of solution mappings for quasiequilibrium problems, stability of solution mappings for bilevel equilibrium problems, the Levitin-Polyak well-posedness for bilevel equilibrium problems and Painlevé-Kuratowski convergence of solution sets for quasiequilibrium problems. Moreover, several special cases of optimization related problems such as quasivariational inequalities of the Minty type and the Stampacchia type, variational inequality problems with equilibrium constraints, optimization problems with equilibrium constraints and traffic network problems with equilibrium constraints are also discussed.

3 Objective of the research

Study objects of this thesis are optimization related problems such as quasiequilibrium problems, quasivariational inequalities of the Minty type and the Stampacchia type, bilevel equilibrium problems, variational inequality problems with equilibrium constraints, optimization problems with equilibrium constraints and traffic network problems with equilibrium constraints.

4 Scope of the research

The thesis is concerned with study the Levitin-Polyak well-posedness, stability and Painlevé-Kuratowski convergence of solutions for optimization related problems.

5 Methodology of the research

We use the theoretical study method of functional analysis, the method of the variational analysis and optimization theory in process of studying the topic.

6 Contribution of the thesis

The results of thesis contribute more abundant for the researching directions of Levitin-Polyak well-posedness, stability and Painlevé-Kuratowski convergence in optimization theory.

The thesis can be a reference for under graduated students, master students and doctoral students in analysis major in general, and the optimization theory and applications in particular.

7 Overview and Organization of the research

Besides the sections of usual notations, preface, general conclusions and recommendations, list of the author's articles related to the thesis and references, the thesis is organized into three chapters.

Chapter 1 presents the parametric strong vector quasiequilibrium problems in Hausdorff topological vector spaces. In section 1.3, we introduce parametric gap functions for these problems, and study the continuity property of these functions. In section 1.4, we present two key hypotheses related to the gap functions for the considered problems and also study characterizations of these hypotheses. Afterwards, we prove that these hypotheses are not only sufficient but also necessary for the Hausdorff lower semicontinuity and Hausdorff continuity of solution mappings to these problems. In section 1.5, as applications, we derive several results on Hausdorff (lower) continuity properties of the solution mappings in the special cases of variational inequalities of the Minty type and the Stampacchia type.

Chapter 2 presents the vector quasiequilibrium problems under perturbation in terms of suitable asymptotically solving sequences, not embedding given problems into a parameterized family. In section 2.1, we introduce gap functions for these problems and study the continuity property of these functions. In section 2.2, by employing some types of convergences for mapping and set sequences, we obtain the Painlevé-Kuratowski upper convergence of solution sets for the reference problems. Then, by using nonlinear scalarization functions, we propose gap functions for such problems, and later employing these functions, we study necessary and sufficient conditions for Painlevé-Kuratowski lower convergence and Painlevé-Kuratowski convergence. In section 2.3, as an application, we

discuss the special case of vector quasivariational inequality.

Chapter 3 presents the stability of solutions and Levitin-Polyak well-posedness for bilevel vector equilibrium problems. In section 3.1, we studty the stability of solutions for parametric bilevel vector equilibrium problems in Hausdorff topological vector spaces. Then we study the stability conditions such as (Hausdorff) upper semicontinuity and (Hausdorff) lower semicontinuity of solutions for such problems. Many examples are provided to illustrate the essentialness of the imposed assumptions. For the applications, we obtain the stability results for the parametric vector variational inequality problems with equilibrium constraints and parametric vector optimization problems with equilibrium constraints. In section 3.2, we introduce the concepts of Levitin-Polyak well-posedness and Levitin-Polyak well-posedness in the generalized sense for strong bilevel vector equilibrium problems. The notions of upper/lower semicontinuity involving variable cones for vector-valued mappings and their properties are proposed and studied. Using these generalized semicontinuity notions, we investigate sufficient and/or necessary conditions of the Levitin-Polyak well-posedness for the reference problems. Some metric characterizations of these Levitin-Polyak well-posedness concepts in the behavior of approximate solution sets are also discussed. As an application, we consider the special case of traffic network problems with equilibrium constraints.

CHAPTER 1

CONTINUITY OF SOLUTION MAPPINGS FOR QUASIEQUILIBRIUM PROBLEMS

In this chapter, we present the continuity of solution mappings of parametric strong vector quasiequilibrium problems. Firstly, we consider parametric quasiequilibrium problems and recall some preliminary results which are needed in the sequel. Afterward, we introduce parametric gap functions for these problems, and study the continuity property of these functions. Next, we present two key hypotheses related to the gap functions for the considered problems and also study characterizations of these hypotheses. Then, we prove that these hypotheses are not only sufficient but also necessary for the Hausdorff lower semicontinuity and Hausdorff continuity of solution mappings to these problems. Finally, as applications, we derive several results on Hausdorff (lower) continuity properties of the solution mappings in the special cases of variational inequalities of the Minty type and the Stampacchia type.

1.1 Preliminaries

Definition 1.1.3. Let X and Y be two topological Hausdorff spaces and $F : X \rightrightarrows Y$ be a multifunction.

(i) F is said to be *upper semicontinuous (usc) at* x_0 if for each open set $U \supset F(x_0)$, there is a neighborhood V of x_0 such that $U \supset F(x)$, for all $x \in V$.

(ii) F is said to be *lower semicontinuous (lsc) at* x_0 if $F(x_0) \cap U \neq \emptyset$ for some open set $U \subset Y$ implies the existence of a neighborhood V of x_0 such that $F(x) \cap U \neq \emptyset$, for all $x \in V$.

(iii) F is said to be *continuous at* x_0 if it is both lsc and usc at x_0.

(iv) F is said to be closed at $x_0 \in \mathrm{dom}F$ if for each net $\{(x_\alpha, z_\alpha)\} \subset \mathrm{graph}F$ such that $(x_\alpha, z_\alpha) \to (x_0, z_0)$, it follows that $(x_0, z_0) \in \mathrm{graph}F$.

Definition 1.1.4. Let X and Y be two topological Hausdorff vector spaces and $F : X \rightrightarrows Y$ be a multifunction.

(i) F is said to be *Hausdorff upper semicontinuous (H-usc) at* x_0 if for each neighborhood U of the origin in Y, there exists a neighborhood V of x_0 such that, $F(x) \subset F(x_0) + U, \forall x \in V$.

(ii) F is said to be *Hausdorff lower semicontinuous (H-lsc) at* x_0 if for each neighborhood U of the origin in Y, there exists a neighborhood V of x_0 such that $F(x_0) \subset F(x) + U, \forall x \in V$.

(iii) F is said to be *H-continuous at* x_0 if it is both H-lsc and H-usc at x_0.

We say that F satisfies a certain property on a subset $A \subset X$ if F satisfies it at every point of A. If $A = X$, we omit "on X" in the statement.

Lemma 1.1.8. For any fixed $e \in \text{intC}, y \in Y$ and the nonlinear scalarization function $\xi_e : Y \to R$ defined by $\xi_e(y) := \min\{r \in R : y \in re - C\}$, we have

(i) ξ_e is a continuous and convex function on Y;

(ii) $\xi_e(y) \leq r \Leftrightarrow y \in re - C$;

(iii) $\xi_e(y) > r \Leftrightarrow y \notin re - C$.

1.2 Quasiequilibrium problems

Let X, Y, Z, P be Hausdorff topological vector spaces, $A \subset X$, $B \subset Y$ and $\Gamma \subset P$ be nonempty subsets, and let C be a closed convex cone in Z with int$C \neq \emptyset$. Let $K : A \times \Gamma \rightrightarrows A, T : A \times \Gamma \rightrightarrows B$ be multifunctions and $f : A \times B \times A \times \Gamma \to Z$ be an equilibrium function, i.e., $f(x, t, x, \gamma) = 0$ for all $x \in A, t \in B, \gamma \in \Gamma$. Motivated and inspired by variational inequalities in the sense of Minty and Stampacchia, we consider the following two parametric strong vector quasiequilibrium problems.

(QEP$_1$) finding $x \in K(x, \gamma)$ such that

$$f(x, t, y, \gamma) \in C, \forall y \in K(x, \gamma), \forall t \in T(y, \gamma).$$

(QEP$_2$) finding $x \in K(x, \gamma)$ and $t \in T(x, \gamma)$ such that

$$f(x, t, y, \gamma) \in C, \forall y \in K(x, \gamma).$$

For each $\gamma \in \Gamma$, we denote the solution sets of (QEP$_1$) and (QEP$_2$) by $S_1(\gamma)$ and $S_2(\gamma)$, respectively.

1.3 Gap functions for (QEP$_1$) and (QEP$_2$)

In this section, we introduce the parametric gap functions for (QEP$_1$) and (QEP$_2$).

Definition 1.3.1. A function $g : A \times \Gamma \to \mathbb{R}$ is said to be a *parametric gap function* for problem (QEP$_1$) ((QEP$_2$), respectively), if:

(a) $g(x, \gamma) \geq 0$, for all $x \in K(x, \gamma)$;

(b) $g(x, \gamma) = 0$ if and only if $x \in S_1(\gamma)$ ($x \in S_2(\gamma)$, respectively.)

Now we suppose that K and T have compact valued in a neighborhood of the reference point. We define two functions $p : A \times \Gamma \to R$ and $h : A \times \Gamma \to R$ as follows

$$p(x, \gamma) = \max_{t \in T(y,\gamma)} \max_{y \in K(x,\gamma)} \xi_e(-f(x, t, y, \gamma)), \tag{1.1}$$

and

$$h(x, \gamma) = \min_{t \in T(x,\gamma)} \max_{y \in K(x,\gamma)} \xi_e(-f(x, t, y, \gamma)). \tag{1.2}$$

Since $K(x, \gamma)$ and $T(x, \gamma)$ are compact sets for any $(x, \gamma) \in A \times \Gamma$, ξ_e and f are continuous, p and h are well-defined.

Theorem 1.3.2.

(i) *The function $p(x, \gamma)$ defined by* (1.1) *is a parametric gap function for problem* (QEP$_1$).

(ii) *The function $h(x, \gamma)$ defined by* (1.2) *is a parametric gap function for problem* (QEP$_2$).

Theorem 1.3.4. *Consider* (QEP$_1$) *and* (QEP$_2$), *assume that K and T are continuous with compact values on $A \times \Gamma$. Then, p and h are continuous on $A \times \Gamma$.*

1.4 Continuity of solution mappings for (QEP$_1$) and (QEP$_2$)

In this section, we establish the Hausdorff lower semicontinuity and Hausdorff continuity of the solution mappings to (QEP$_1$) and (QEP$_2$).

Theorem 1.4.1. *Consider* (QEP$_1$) *and* (QEP$_2$), *assume that A is compact, K is continuous with compact values on A, and* $\mathrm{L}_{\geq_C 0} f$ *is closed. Then,*

(i) *S_1 is both upper semicontinuous and closed with compact values on Γ if T is lower semicontinuous on A,*

(ii) *S_2 is both upper semicontinuous and closed with compact values on Γ if T is upper semicontinuous with compact values on A,*

where $\mathrm{L}_{\geq_C 0} f = \{(x,t,y,\gamma) \in X \times Z \times X \times \Gamma \mid f(x,t,y,\gamma) \in C\}$.

Motivated by the hypotheses (H_1) in Zhao (in 1997), we introduce the following key assumptions.

$(H_p(\gamma_0))$: Given $\gamma_0 \in \Gamma$. For any open neighborhood U of the origin in X, there exist $\rho > 0$ and a neighborhood $V(\gamma_0)$ of γ_0 such that for all $\gamma \in V(\gamma_0)$ and $x \in E(\gamma) \setminus (S_1(\gamma) + U)$, one has $p(x,\gamma) \geq \rho$.

$(H_h(\gamma_0))$: Given $\gamma_0 \in \Gamma$. For any open neighborhood U of the origin in X, there exist $\rho > 0$ and a neighborhood $V(\gamma_0)$ of γ_0 such that for all $\gamma \in V(\gamma_0)$ and $x \in E(\gamma) \setminus (S_2(\gamma) + U)$, one has $h(x,\gamma) \geq \rho$.

Now, we show that the hypotheses $(H_p(\gamma_0))$ and $(H_h(\gamma_0))$ are not only sufficient but also necessary for the Hausdorff lower semicontinuity and Hausdorff continuity of the solution mappings to (QEP_1) and (QEP_2), respectively.

Theorem 1.4.6. *Consider* (QEP_1) *and* (QEP_2)*, suppose that* A *is compact,* K *and* T *are continuous with compact values in* $A \times \Gamma$*,* f *is continuous in* $A \times B \times A \times \Lambda$*. Then,*

(i) S_1 *is Hausdorff lower semicontinuous on* Γ *if and only if* $(H_p(\gamma_0))$ *is satisfied,*

(ii) S_2 *is Hausdorff lower semicontinuous on* Γ *if and only if* $(H_h(\gamma_0))$ *is satisfied..*

Theorem 1.4.7. *Suppose that all the conditions in Theorem 1.4.6 are satisfied. Then,*

(i) S_1 *is Hausdorff continuous with compact values in* Γ *if and only if* $(H_p(\gamma_0))$ *holds,*

(ii) S_2 *is Hausdorff continuous with compact values in* Γ *if and only if* $(H_h(\gamma_0))$ *holds.*

1.5 Application to quasivariational inequality problems

Let X, Y, Z, A, B, C, K, T be as in Sect. 2, $L(X;Y)$ be the space of all linear continuous operators from X into Y and $g : A \times \Lambda \to A$ be a vector function. $\langle t, x\rangle$ denotes the value of a linear operator $t \in L(X;Y)$ at $x \in X$. For each $\gamma \in \Gamma$, we consider the following two parametric strong vector quasivariational inequalities of the types of Minty and Stampacchia (in short, (MQVI) and (SQVI), respectively).

(MQVI) finding $x \in K(x,\gamma)$ such that

$$\langle t, y - g(x,\gamma)\rangle \in C, \forall y \in K(x,\gamma), \forall t \in T(y,\gamma).$$

(SQVI) finding $x \in K(x,\gamma)$ and $t \in T(x,\gamma)$ such that

$$\langle t, y - g(x,\gamma)\rangle \in C, \forall y \in K(x,\gamma).$$

By setting

$$f(x,t,y,\gamma) = \langle t, y - g(x,\gamma)\rangle, \tag{1.3}$$

the problems (MQVI) and (SQVI) become special cases of (QEP_1) and (QEP_2), respectively. For each $\gamma \in \Gamma$, we denote the solution sets of the problems (MQVI) and (SQVI) by $\Phi(\gamma)$ and $\Psi(\gamma)$, respectively.

The following results are derived from the main results of Section 1.4.

Corollary 1.5.1. *Consider* (MQVI) *and* (SQVI)*, assume that A is compact, K and T are continuous with compact values in $A \times \Gamma$, and g is continuous in $A \times \Gamma$. Then,*

(i) *Φ is Hausdorff lower semicontinuous on Γ if and only if $(H_p(\gamma_0))$ holds,*

(ii) *Ψ is Hausdorff lower semicontinuous on Γ if and only if $(H_h(\gamma_0))$ holds.*

Corollary 1.5.3. *Suppose that all the conditions in Corollary 1.5.1 are satisfied. Then,*

(i) *Φ is Hausdorff continuous with compact values in Γ if and only if $(H_p(\gamma_0))$ holds,*

(ii) *Ψ is Hausdorff continuous with compact values in Γ if and only if $(H_h(\gamma_0))$ holds.*

Conclusions of Chapter 1

In this chapter, we obtained the following main results:

- Give some gap functions for problems (QEP_1) and (QEP_2) (Denifition 1.3.1 and Theorem 1.3.2). Then, establish continuity property of these functions (Theorem 1.3.4).

- Establish upper semicontinuity of solution mappings for problems (QEP_1) and (QEP_2) (Theorem 1.4.1). Base on the gap functions, we study two key hypotheses $(H_p(\gamma_0))$ and $(H_h(\gamma_0))$. Afterwards, we prove that these hypotheses are not only sufficient but also necessary for the Hausdorff lower semicontinuity and Hausdorff continuity of solution mappings to these problems (Theorem 1.4.6 and Theorem 1.4.7).

- From the main results in Section 1.3, we derive several results on Hausdorff (lower) continuity properties of the solution mappings in the special cases of variational inequalities of the Minty type and the Stampacchia type (Corollary 1.5.1 and Corollary 1.5.3).

These results were published in the article:

L. Q. Anh and N. V. Hung (2018), Gap functions and Hausdorff continuity of solution mappings to parametric strong vector quasiequilibrium problems, *Journal of Industrial and Management Optimization*, **14**, 65-79.

CHAPTER 2

CONVERGENCE OF SOLUTION SETS FOR QUASIEQUILIBRIUM PROBLEMS

In this chapter, we consider vector quasiequilibrium problems under perturbation in terms of suitable asymptotically solving sequences, not embedding given problems into a parameterized family. By employing some types of convergences for mapping and set sequences, we obtain the Painlevé-Kuratowski upper convergence of solution sets for the reference problems. Then, using nonlinear scalarization functions, we propose gap functions for such problems, and later employing these functions, we study necessary and sufficient conditions for Painlevé-Kuratowski lower convergence and Painlevé-Kuratowski convergence. As an application, we discuss the special case of vector quasivariational inequality.

2.1 Sequence of quasiequilibrium problems

Let X, Y, Z be metric linear spaces, $A \subset X, B \subset Y$ be nonempty compact subsets. Recall that $\mathcal{E}$ is called a metric linear space iff it is both a metric space and a linear space and the metric d of $\mathcal{E}$ is translation invariant. Let $K : A \rightrightarrows A, T : B \rightrightarrows B$ be set-valued mappings and $f : A \times B \times A \to Z$ be a single-valued mapping. Let $C : A \rightrightarrows Z$ be a set-valued mapping such that for each $x \in A$, $C(x)$ is a proper, closed and convex cone in Z with $\text{int}C(x) \neq \emptyset$.

We consider the following generalized vector quasiequilibrium problem.
(WQEP) finding $\bar{x} \in K(\bar{x})$ and $\bar{z} \in T(\bar{x})$ such that

$$f(\bar{x}, \bar{z}, y) \in Y \setminus -\text{int}C(\bar{x}), \forall y \in K(\bar{x}).$$

For sequences of set-valued mappings $K_n : A \rightrightarrows A, T_n : A \rightrightarrows Y$, and single-valued mappings $f_n : A \times B \times A \to Z$, for $n \in \mathbb{N} \setminus \{0\}$, we consider the following sequence of generalized vector quasiequilibrium problems.
$(\text{WQEP})_n$ finding $\bar{x} \in K_n(\bar{x})$ and $\bar{z} \in T_n(\bar{x})$ such that

$$f_n(\bar{x}, \bar{z}, y) \in Y \setminus -\text{int}C(\bar{x}), \forall y \in K_n(\bar{x}).$$

We denote the solution sets of problems (WQEP) and $(\text{WQEP})_n$ by $S(f,T,K)$ and $S(f_n,T_n,K_n)$, respectively (resp).

Definitions 2.1.1. A sequence of sets $\{D_n\}$, $D_n \subset X$, is said to *upper converge* (*lower converge*) in the sense of Painlevé-Kuratowski to D if $\limsup_{n\to\infty} D_n \subset D$ ($D \subset \liminf_{n\to\infty} D_n$, resp). $\{D_n\}$ is said to *converge* in the sense of Painlevé-Kuratowski to D if $\limsup_{n\to\infty} D_n \subset D \subset \liminf_{n\to\infty} D_n$. The set-valued mapping G is said to be continuous at x_0 if is both outer semicontinuous and inner semicontinuous at x_0.

Definitions 2.1.2. A sequence of sets $\{D_n\}$, $D_n \subset X$, is said to *upper converge* (*lower converge*) in the sense of Painlevé-Kuratowski to D if $\limsup_{n\to\infty} D_n \subset D$ ($D \subset \liminf_{n\to\infty} D_n$, resp). $\{D_n\}$ is said to *converge* in the sense of Painlevé-Kuratowski to D if $\limsup_{n\to\infty} D_n \subset D \subset \liminf_{n\to\infty} D_n$. The set-valued mapping G is said to be continuous at x_0 if is both outer semicontinuous and inner semicontinuous at x_0.

Definitions 2.1.3. A sequence of mappings $\{f_n\}$, $f_n : X \to Y$, is said to *converge continuously* to a mapping $f : X \to Y$ at x_0 if $\lim_{n\to\infty} f_n(x_n) = f(x_0)$ for any $x_n \to x_0$.

Definitions 2.1.4. Let $\{G_n\}$, $G_n : X \rightrightarrows Y$, be a sequence of set-valued mappings and $G : X \rightrightarrows Y$ be a set-valued mapping. $\{G_n\}$ is said to *outer-converge continuously* (*inner-converge continuously*) to G at x_0 if $\limsup_{n\to\infty} G_n(x_n) \subset G(x_0)$ ($G(x_0) \subset \liminf_{n\to\infty} G_n(x_n)$, resp) for any $x_n \to x_0$. $\{G_n\}$ is said to *converge continuously* to G at x_0 if $\limsup_{n\to\infty} G_n(x_n) \subset G(x_0) \subset \liminf_{n\to\infty} G_n(x_n)$ for any $x_n \to x_0$.

Lemma 2.1.5. *Let X and Z be convex Hausdorff topological vector spaces, and let $C : X \rightrightarrows Z$ be a set-valued mapping such that $C(x)$ is a proper, closed and convex cone in Z with $\text{int}C(x) \neq \emptyset$ for all $x \in X$. Furthermore, let $e : X \to Z$ be the continuous selection of the set-valued mapping $\text{int}C(.)$. Consider a set-valued mapping $V : X \rightrightarrows Z$ given by $V(x) := Z \setminus \text{int}C(x)$ for all $x \in X$. The nonlinear scalarization function $\xi_e : X \times Z \to \mathbb{R}$ defined by $\xi_e(x,y) := \inf\{r \in \mathbb{R} \mid y \in re(x) - C(x)\}$ for all $(x,y) \in X \times Z$ satisfies following properties:*

(i) $\xi_e(x,y) < r \Leftrightarrow y \in re(x) - \text{int}C(x)$;

(ii) $\xi_e(x,y) \geq r \Leftrightarrow y \notin re(x) - \text{int}C(x)$;

(iii) *If V and C are upper semicontinuous, then ξ_e is continuous.*

Definition 2.1.6. A function $q : A \to \mathbb{R}$ is said to be a gap function for problem (WQEP) ((WQEP$_n$), respectively), if:

(a) $q(x) \geq 0$, for all $x \in K(x)$;

(b) $q(x) = 0$ if and only if $x \in S(f,T,K)$ ($x \in S(f_n,T_n,K_n)$, respectively.)

Suppose that K, K_n, T, T_n are compact-valued, and f, f_n are continuous. For simplicity's sake, we denote $K_0 := K$, $T_0 := T$, $f_0 := f$. For $n \in \mathbb{N}$, functions $h_n : A \to \mathbb{R}$ given by

$$h_n(x) = \min_{z \in T_n(x)} \max_{y \in K_n(x)} \{-\xi_e(x, f_n(x, z, y))\} \tag{2.1}$$

are well-defined. In the sequel, we assume further that f_n are equilibrium mappings, i.e., $f_n(x, z, x) = 0$ for all $x \in A$ and $n \in \mathbb{N}$.

Proposition 2.1.7. *For each $n \in \mathbb{N}$, the function $h_n(x)$ defined by* (2.1) *is a gap function for problem* (WQEP$_n$).

Proposition 2.1.8. *For $n \in \mathbb{N}$, assume that*

(i) *K_n and T_n are continuous and compact-valued*;

(ii) *V, C are upper semicontinuous and e is continuous.*

Then, h_n defined by (2.1) are continuous.

Proposition 2.1.9. *For $n \in N$, assume that*

(i) *K_n are continuous and compact-valued*;

(ii) *T_n are upper semicontinuous and compact-valued*;

(iii) *W is closed.*

Then, $S(f_n, T_n, K_n)$ are compact.

2.2 Convergence of solution sets for equilibrium problems

In this section, we study the convergence of the solutions for (WQEP) and (WQEP$_n$).

Theorem 2.2.1. *Consider* (WQEP) *and* (WQEP$_n$), *assume that*

(i) *$\{K_n\}$ converges continuously to K*;

(ii) *$\{T_n\}$ outer converges continuously to T*;

(iii) *$\{f_n\}$ converges continuously to f*;

(iv) *W is closed.*

Then, $\limsup_{n\to\infty} S(f_n, T_n, K_n) \subset S(f, T, K)$.

Motivated by the hypothesis (H_1) of Zhao (in 1997), we introduce the following key hypothesis and employ it to study the Painlevé-Kuratowski convergence of the solution sets for (WQEP) and (WQEP$_n$).

(H_h): For any neighborhood U of the origin in X, there exist $\alpha \in (0, +\infty)$ and $n_0 \in \mathbb{N}$ such that $h_n(x) \geq \alpha$ for all $n \geq n_0$ and $x \in K_n(x) \setminus (S(f_n, T_n, K_n) + U))$.

Theorem 2.2.12. *Consider* (WQEP) *and* (WQEP$_n$)*, impose all assumptions of Proposition 2.1.9 and assume further that*

(i) $\{K_n\}$ *converges continuously to* K;

(ii) $\{T_n\}$ *converges continuously to* T;

(iii) $\{f_n\}$ *converges continuously to* f;

(iv) V, C *are upper semicontinuous.*

Then, $S(f, T, K) \subset \liminf_{n\to\infty} S(f_n, T_n, K_n)$ *if and only if* (H_h) *holds.*

Theorem 2.2.13. *Assume that all assumptions of Theorem 2.2.12 are satisfied. Then,* $S(f_n, T_n, K_n)$ *converge to* $S(f, T, K)$ *in the sense of Painlevé - Kuratowski if and only if* (H_h) *holds.*

2.3 Application to quasivariational inequality

Let X, Z be Banach spaces, $Y = L(X, Z)$, the space of all linear continuous operators from X into Z, A, B, C, K, T, K_n, T_n be as in Sect. 2.1. Denoted by $\langle z, x\rangle$ the value of a linear operator $z \in L(X, Y)$ at $x \in X$. Then, we consider the generalized vector quasivariational inequalities

(QVI) Finding $\bar{x} \in K(\bar{x})$ and $\bar{z} \in T(\bar{x})$ such that

$$\langle \bar{z}, y - \bar{x}\rangle \in Y \setminus -\mathrm{int}C(\bar{x}), \forall y \in K(\bar{x}).$$

(QVI)$_n$ Finding $\bar{x} \in K_n(\bar{x})$ and $\bar{z} \in T_n(\bar{x})$ such that

$$\langle \bar{z}, y - \bar{x}\rangle \in Y \setminus -\mathrm{int}C(\bar{x}), \forall y \in K_n(\bar{x}).$$

We denote the solution sets of (QVI) and (QVI)$_n$ by $S(T, K)$ and $S(T_n, K_n)$, resp. By setting $f(x, z, y) = f_n(x, y, z) = \langle z, y-x\rangle$, then (QVI) becomes a special case of (WQEP).

By applying Theorem 2.2.1, we obtain the following result.

Corollary 2.3.1. *Consider* (QVI) *and* (QVI)$_n$*, assume that*

(i) $\{K_n\}$ *converges continuously to* K;

(ii) $\{T_n\}$ *outer converges continuously to* T;

(iii) W *is closed.*

Then, $\limsup_{n\to\infty} S(T_n, K_n) \subset S(T, K)$.

Corollary 2.3.1. *Consider* (QVI) *and* $(\text{QVI})_n$*, assume that*

(i) $\{K_n\}$ *converges continuously to* K;

(ii) $\{T_n\}$ *outer converges continuously to* T;

(iii) W *is closed.*

Then, $\limsup_{n\to\infty} S(T_n, K_n) \subset S(T, K)$.

For the lower convergence in the sense of Painlevé - Kuratowski for (QVI), we will apply Theorem 2.2.12 to such problems.

Corollary 2.3.2. *For* $n \in \mathbb{N}$*, consider* (QVI) *and* $(\text{QVI})_n$ *and assume that*

(i) K_n *are continuous and compact-valued, and* $\{K_n\}$ *converges continuously to* K;

(ii) T_n *are upper semicontinuous and compact-valued, and* $\{T_n\}$ *converges continuously to* T;

(iii) V, C *are upper semicontinuous*;

(iv) W *is closed.*

Then, $S(T, K) \subset \liminf_{n\to\infty} S(T_n, K_n)$ *if only if* (H_h) *holds.*

Corollary 2.3.4. *Impose all assumptions of Corollary 2.3.2. Then,* $S(T_n, K_n)$ *converge to* $S(T, K)$ *in the sense of Painlevé - Kuratowski if only if* (H_h) *holds.*

Conclusions of Chapter 2

In this chapter, we obtained the following main results

- Give gap function sequences for problems (WQEP) and $(\text{WQEP})_n$ (Proposition 2.1.7). Then, establish continuity property of these functions (Proposition 2.1.8).

- Establish Painlevé-Kuratowski upper convergence of solution sets for the reference problems (Theorem 2.2.1). Base on the gap function sequences, we study the key hypotheses (H_h). Afterwards, we study necessary and sufficient conditions for Painlevé-Kuratowski lower convergence and Painlevé-Kuratowski convergence (Theorem 2.2.12 and Theorem 2.2.13).

- As an application, we discuss the special case of vector quasivariational inequality (Corollary 2.3.1, Corollary 2.3.2 and Corollary 2.3.4).

These results were published in the article:

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CHAPTER 3

STABILITY AND WELL-POSEDNESS FOR BILEVEL EQUILIBRIUM PROBLEMS.

In this chapter, we study stability of solutions and Levitin-Polyak well-posedness for bilevel vector equilibrium problems. Firstly, we studty the (Hausdorff) upper semicontinuity and (Hausdorff) lower semicontinuity of solutions for parametric bilevel vector equilibrium problems. For the applications, we obtain the stability results for the parametric vector variational inequality problems with equilibrium constraints and parametric vector optimization problems with equilibrium constraints. Secondly, we introduce the concepts of Levitin-Polyak well-posedness and Levitin-Polyak well-posedness in the generalized sense for strong bilevel vector equilibrium problems. Then, we investigate sufficient and/or necessary conditions of the Levitin-Polyak well-posedness for the reference problems. Some metric characterizations of these Levitin-Polyak well-posedness concepts in the behavior of approximate solution sets are also discussed. As an application, we consider the special case of traffic network problems with equilibrium constraints.

3.1 Stability of solution mappings for bilevel equilibrium problems

Let X, Y, Z be Hausdorff topological vector spaces. A and Λ are nonempty convex subsets of X and Y, respectively, and $C \subset Z$ is a solid pointed closed convex cone. Let $K_{1,2} : A \times \Lambda \rightrightarrows A$ be two multifunctions, and $f : A \times A \times \Lambda \to Z$ be a vector function. For each $\lambda \in \Lambda$, we consider the following parametric vector quasiequilibrium problem:
(SQEP) Find $\bar{x} \in K_1(\bar{x}, \lambda)$ such that

$$f(\bar{x}, y, \lambda) \in C, \forall y \in K_2(\bar{x}, \lambda).$$

For each $\lambda \in \Lambda$, let $E(\lambda) = \{x \in A \mid x \in K_1(x, \lambda)\}$ and we denote the solution set of (SQEP) by $S(\lambda)$, i.e., $S(\lambda) = \{x \in K_1(x, \lambda) \mid f(x, y, \lambda) \in C, \forall y \in K_2(x, \lambda)\}$.

Let W be a Hausdorff topological vector space, and Γ be a nonempty subset of W. Let $B = A \times \Lambda$ and $h : B \times B \times \Gamma \to Z$ be a vector function, $C' \subset Z$ be a solid pointed closed convex cone. We consider the following parametric bilevel vector equilibrium problem:

(BEP) finding $\bar{x}^* \in \text{graph}S^{-1}$ such that

$$h(\bar{x}^*, y^*, \gamma) \in C', \forall y^* \in \text{graph}S^{-1},$$

where $\text{graph}S^{-1} = \{(x, \lambda) \mid x \in S(\lambda)\}$ is the graph of S^{-1}.

For each $\gamma \in \Gamma$, we denote the solution set of (BEP) by $\Phi(\gamma)$, and we assume that $\Phi(\gamma) \neq \emptyset$ for each γ in a neighborhood of the reference point.

For a multifunction $G : X \rightrightarrows Z$ between two linear spaces, G is said to be convex (concave) on a convex subset $A \subset X$ if, for each $x_1, x_2 \in A$ and $t \in [0, 1]$,

$$tG(x_1) + (1-t)G(x_2) \subset G(tx_1 + (1-t)x_2)$$

$$(G(tx_1 + (1-t)x_2) \subset tG(x_1) + (1-t)G(x_2), \text{respectively}).$$

Let $\varphi : X \to Z$ be a vector function and $C \subset Z$ be a solid pointed closed convex cone. For $\theta \in Z$, we use the following notations for level sets of φ with respect to C, for different ordering cones (by the context, no confusion occurs).

$$\begin{aligned} L_{\geq_C \theta}\varphi :=& \{x \in X \mid \varphi(x) \in \theta + C\}, \\ L_{\not>_C \theta}\varphi :=& \{x \in X \mid \varphi(x) \notin \theta + \text{int}C\}, \end{aligned}$$

and similarly for other level sets $L_{\leq_C \theta}\varphi, L_{\not<_C \theta}\varphi, L_{\not\geq_C \theta}\varphi, L_{>_C \theta}\varphi$, etc.

Now, we discuss the upper semicontinuity of the solutions for problem (BEP).

Theorem 3.1.1. *Consider* (BEP)*, assume that* Λ *is compact and the following conditions hold:*

(i) *E is usc with compact values, and K_2 is lsc;*

(ii) *$L_{\geq_C 0}f$ is closed on $A \times A \times \Lambda$;*

(iii) *$L_{\geq_{C'} 0}h$ is closed on $B \times B \times \{\gamma_0\}$.*

Then Φ is both usc and closed at γ_0.

Theorem 3.1.5. *Theorem 3.1.1 is still valid if assumption* (i) *is replaced by*

(i') *A is compact, K_1 is closed, and K_2 is lsc.*

For each $\gamma \in \Gamma$, we consider the following an auxiliary subset of Φ:

$$\begin{aligned} \Phi^*(\gamma) = \{x^* \in \text{graph}S^{-1} \mid & f(x, y, \lambda) \in \text{int}C, h(x^*, y^*, \gamma) \in \text{int}C', \\ & \forall y \in K_2(x, \lambda), \forall y^* \in \text{graph}S^{-1}\}. \end{aligned}$$

Definition 3.1.7. Let X, Z be Hausdorff topological vector spaces, $\varphi : X \to Z$ be a vector function, and $C \subset Z$ be a solid pointed closed convex cone. The function φ is

said to be generalized C-quasiconcave in a nonempty convex subset $A \subset X$, if for each $x_1, x_2 \in A$, from $\varphi(x_1) \in C$ and $\varphi(x_2) \in \text{int}C$, it follows that, for each $t \in (0,1)$,

$$\varphi(tx_1 + (1-t)x_2) \in \text{int}C.$$

Theorem 3.1.8. *Consider* (BEP)*, assume that* Λ *is compact and the following conditions hold:*

(i) *E is convex and continuous with compact values, K_2 is concave and continuous with compact values;*

(ii) *$L_{\not>_C 0}f$, $L_{\geq_C 0}f$ are are closed on $A \times A \times \Lambda$ and $L_{\not>_{C'} 0}h$ is closed on $B \times B \times \{\gamma_0\}$;*

(iii) *f is generalized C-quasiconcave;*

(iv) *$h(\cdot, \cdot, y^*, \gamma_0)$ is generalized C'-quasiconcave.*

Then Φ is lower semicontinuous at γ_0.

Passing to the Hausdorff lower semicontinuity, continuity and Hausdorff continuity of the solution mapping for problem (BEP), we obtain the following result.

Theorem 3.1.12. *Impose all the assumptions of Theorem 3.1.8 and assume further that*

(v) *$L_{\geq_{C'} 0}h(\cdot, \cdot, y^*, \gamma_0)$ is closed on B.*

Then Φ is Hausdorff lower semicontinuous at γ_0.

Theorem 3.1.14.

(i) *Suppose that all the assumptions of Theorem 3.1.8 are satisfied. Then Φ is continuous at γ_0, if the conditions of Theorem 3.1.1 or that of Theorem 3.1.5 hold.*

(ii) *Suppose that all the assumptions of Theorem 3.1.12 are satisfied. Then Φ is Hausdorff continuous at γ_0, if the conditions of Theorem 3.1.1 or that of Theorem 3.1.5 hold.*

Now, we discuss only some results for two important special cases of (BEP). Firstly, we consider variational inequality with equilibrium constraints. Let $X, Y, Z, W, C, C', A, B, \Gamma, \Lambda, K_1, K_2, f$ be as in problem (BEP), and let $L(X \times Y, Z)$ be the space of all linear continuous operators from $X \times Y$ into Z, and $T : \Gamma \times B \to L(X \times Y, Z)$ be a vector function. $\langle z, x \rangle$ denotes the value of a linear operator $z \in L(X \times Y; Z)$ at $x \in B$. For each $\gamma \in \Gamma$, we consider the following parametric vector variational inequality with equilibrium constraints: (VIEC) finding $\bar{x}^* \in \text{graph}S^{-1}$ such that

$$\langle T(\bar{x}^*, \gamma), y^* - \bar{x}^* \rangle \in C', \forall y^* \in \text{graph}S^{-1},$$

where S is the solution mapping of problem (SQEP).

Setting $h(x^*, y^*, \gamma) = \langle T(x^*, \gamma), y^* - x^* \rangle$, we see that (VIEC) becomes a special case of (BEP). For $\gamma \in \Gamma$, we denote the solution set of (VIEC) by $\Psi(\gamma)$.

The following results are derived from Theorem 3.1.1.

Corollary 3.1.15. *Consider* (VIEC)*, assume that*

(i) *E is usc with compact values, and K_2 is lsc;*

(ii) *$L_{\geq_{C'}0} f$ is closed on $A \times A \times \Lambda$;*

(iii) *the set $\{(x^*, y^*, \gamma) \mid \langle T(x^*, \gamma), y^* - x^* \rangle \in C'\}$ is closed on $B \times B \times \{\gamma_0\}$.*

Then Ψ is both upper semicontinuous and closed at γ_0.

Secondly, we consider optimization problems with equilibrium constraints. Let $X, Y, Z, W, A, B, C, C', \Lambda, \Gamma, K_1, K_2, f$ be as in problem (BEP), and let $g : B \times \Lambda \to Z$ be a vector function. For each $\gamma \in \Gamma$, we consider the following parametric vector optimization problem with equilibrium constraints:
(OPEC) finding $\bar{x}^* \in \text{graph} S^{-1}$ such that

$$g(y^*, \gamma) \in g(\bar{x}^*, \gamma) + C', \forall y^* \in \text{graph} S^{-1},$$

where S is the solution mapping of problem (SQEP).

Putting $h(x^*, y^*, \gamma) = g(y^*, \gamma) - g(x^*, \gamma)$, we see that (OPEC) is a special case of (BEP). For $\gamma \in \Gamma$, we denote the solution set of problem (OPEC) by $\Xi(\gamma)$.

Applying Theorem 3.1.1, we obtain the following result.

Corollary 3.1.15. *Consider* (OPEC)*, assume that*

(i) *E is usc with compact values, and K_2 is lsc;*

(ii) *$L_{\geq_{C'}0} f$ is closed on $A \times A \times \Lambda$;*

(iii) *the set $\{(x^*, y^*, \gamma) \mid g(y^*, \gamma) - g(x^*, \gamma) \in C'\}$ is closed on $B \times B \times \{\gamma_0\}$.*

Then Ξ is both upper semicontinuous and closed at γ_0.

3.2 Well-posedness for bilevel equilibrium problems

Let X, W, Z be Banach spaces, A and Λ be nonempty closed subsets of X and W, respectively (resp), and $C_1 : A \rightrightarrows Z$ be a set-valued mapping such that for each $x \in A$, $C_1(x)$ is a pointed, closed and convex cone with $\text{int} C_1(x) \neq \emptyset$, where $\text{int}(\cdot)$ is the interior of $(\cdot)$. For $i = 1, 2$, let $K_i : A \times \Lambda \rightrightarrows A$ be set-valued mappings, and $f : A \times A \times \Lambda \to Z$

be a vector mapping. For $\lambda \in \Lambda$, we consider the following parametric quasi-equilibrium problem. (MSQEP) finding $\bar{x} \in K_1(\bar{x}, \lambda)$ such that

$$f(\bar{x}, y, \lambda) \in C_1(\bar{x}), \forall y \in K_2(\bar{x}, \lambda).$$

For each $\lambda \in \Lambda$, we denote the solution set of (MSQEP) by $S(\lambda)$.

Let Y be a Banach space, $B = A \times \Lambda$, $C_2 : B \rightrightarrows Y$ be a multifunction such that for each $x^* \in B$, $C_2(x^*)$ is a pointed, closed and convex cone with $\text{int}C_2(x^*) \neq \emptyset$, and $h : B \times B \to Y$ be a vector mapping. We consider the following strong bilevel vector equilibrium problem. (MBEP) finding $\bar{x}^* \in \text{graph}S^{-1}$ such that

$$h(\bar{x}^*, y^*) \in C_2(\bar{x}^*), \forall y^* \in \text{graph}S^{-1},$$

where $S(\lambda)$ is the solution set of (MSQEP) and $\text{graph}S^{-1} := \{(x, \lambda) \in A \times \Lambda \mid x \in S(\lambda)\}$.

We denote the solution set of (MBEP) by Ψ, i.e.,

$$\Psi = \{\bar{x}^* = (\bar{x}, \lambda) \in \text{graph}S^{-1} \mid f(\bar{x}, y, \lambda) \in C_1(\bar{x}), \forall y \in K_2(\bar{x}, \lambda) \text{ and } h(\bar{x}^*, y^*) \in C_2(\bar{x}^*),\ \forall y^* = (y, \lambda) \in \text{graph}S^{-1}\}.$$

Picking up ideas from Tanaka (in 1997), we propose the notions of semicontinuity involving variable cone for a vector mapping.

Corollary 3.2.1. Let $C : X \rightrightarrows Z$ be a set-valued mapping such that for each $x \in X$, $C(x)$ is a pointed, closed and convex cone with $\text{int}C(x) \neq \emptyset$. Let $f : X \times \Lambda \to Z$ be a vector function. f is said to be *upper semicontinuous* with respect to C (C-usc) at (x_0, λ_0) if for any neighborhood V of the origin θ_Z in Z, there is a neighborhood U of (x_0, λ_0) such that for all $(x, \lambda) \in U$, $f(x, \lambda) \in f(x_0, \lambda_0) + V - C(x_0)$.

Proposition 3.2.2. *The following conditions are equivalent to each other.*

(a) *f is C-upper semicontinuous.*

(b) *For each $(x_0, \lambda_0) \in X \times \Lambda$ and $d \in \text{int}C(x_0)$, there is a neighborhood U of (x_0, λ_0) such that $f(x, \lambda) \in f(x_0, \lambda_0) + d - \text{int}C(x_0)$ for all $(x, \lambda) \in U$.*

(c) *For each $(x_0, \lambda_0) \in X \times \Lambda$ and $a \in Y$, $f^{-1}(a - \text{int}C(x_0))$ is open.*

Proposition 3.2.3. *Assume that f and g are C-upper semicontinuous and $k \in (0, +\infty)$. Then, (a) $f + g$ is C-upper semicontinuous, (b) kf is C-upper semicontinuous.*

Next, we propose the concepts of Levitin-Polyak well-posedness for bilevel vector equilibrium problems and give some metric characterizations of these concepts. Let $e_1 : A \to Z$ and $e_2 : B \to Y$ be continuous mappings satisfying $e_1(x) \in \text{int}C_1(x)$ and $e_2(x^*) \in \text{int}C_2(x^*)$ for every $x \in A$, and $x^* \in B$, resp.

Definition 3.2.4. A sequence $\{x_n^*\} := \{(x_n, \lambda_n)\}$ is called a Levitin-Polyak (LP) approximating sequence for (MBEP) if

(i) $\{x_n^*\} := \{(x_n, \lambda_n)\} \subset A \times \Lambda, \forall n \in \mathbb{N}$;

(ii) there exists a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+$ converging to 0 such that

$$\begin{aligned} &d(x_n, K_1(x_n, \lambda_n)) \leq \varepsilon_n, \quad \forall n \in \mathbb{N}, \\ &f(x_n, y, \lambda_n) + \varepsilon_n e_1(x_n) \in C_1(x_n), \forall y \in K_2(x_n, \lambda_n), \quad \text{and} \\ &h(x_n^*, y^*) + \varepsilon_n e_2(x_n^*) \in C_2(x_n^*), \forall y^* \in \text{graph} S^{-1}, \end{aligned}$$

where $d(a, M) := \inf_{b \in M} d(a, b)$ is the point-to-set distance.

Definition 3.2.5. The problem (MBEP) is said to be Levitin-Polyak (LP) well-posed if

(i) Ψ is a singleton;

(ii) every LP approximating sequence $\{x_n^*\}$ for (MBEP) converges to the unique solution.

Definition 3.2.6. The problem (MBEP) is said to be Levitin-Polyak (LP) well-posed in the generalized sense if

(i) Ψ is nonempty;

(ii) for every LP approximating sequence $\{x_n^*\}$ for (MBEP), there is a subsequence converging to some point of Ψ.

For $\varepsilon \in \mathbb{R}_+$, the approximate solution set of (MBEP) is given by

$$\begin{aligned} \widetilde{\Psi}(\varepsilon) := \{x^* = (x, \lambda) \in \text{graph} S^{-1} \mid d(x, K_1(x, \lambda)) \leq \varepsilon, \\ f(x, y, \lambda) + \varepsilon e_1(x) \in C_1(x), \forall y \in K_2(x, \lambda), \\ h(x^*, y^*) + \varepsilon e_2(x^*) \in C_2(x^*), \ \forall y^* \in \text{graph} S^{-1}\}. \end{aligned}$$

Theorem 3.2.8. *Consider* (MBEP)*, assume that A and Λ are compact and the following conditions hold*

(i) *K_1 is upper semicontinuous and compact-valued, and K_2 is lower semicontinuous*;

(ii) *f is C_1-upper semicontinuous*;

(iii) *for each $y^* \in graph S^{-1}$, $h(\cdot, y^*)$ is C_2-upper semicontinuous*;

(iv) *C_1 and C_2 are Hausdorff upper semicontinuous.*

Then, $\widetilde{\Psi}$ is upper semicontinuous and compact-valued at 0.

Theorem 3.2.9. *The problem* (MBEP) *is LP well-posed in the generalized sense if and only if $\widetilde{\Psi}$ is upper semicontinuous and nonempty compact-valued at* 0.

Combining Theorems 3.2.8 and 3.2.9, we obtain the relationship between the LP well-posedness and existence solutions of (MBEP).

Theorem 3.2.10. *Assume that all assumptions of Theorem 3.2.8 are satisfied. Then,*

(i) *the problem* (MBEP) *is LP well-posedness in the generalized sense if and only if* Ψ *is nonempty,*

(ii) *the problem* (MBEP) *is LP well-posedness if and only if* Ψ *is a singleton.*

We now present a metric characterization for the LP well-posedness in terms of the behavior of approximate solution sets without the compactness of A and Λ.

Theorem 3.2.11. *Suppose that assumptions* (i)-(iv) *of Theorem 3.2.8 are satisfied. Then, the problem* (MBEP) *is LP well-posed if and only if*

$$\widetilde{\Psi}(\varepsilon) \neq \emptyset, \forall \varepsilon \geq 0 \ \ and \ \ \mathrm{diam}\widetilde{\Psi}(\varepsilon) \to 0 \ \ as \ \ \varepsilon \to 0.$$

Next, we study the traffic network problems with equilibrium constraints as an application. We first recall the model of traffic network problems considered by many authors such as in Wardrop (in 1952), De Luca (in 1995) and Anh and Khanh (in 2010). Consider a transportation network $L = (N, A)$, where N denotes the set of nodes and A denotes the set of arcs. Let $Q = (Q_1, Q_2, ..., Q_n)$ be the set of origin-destination pairs (O/D pairs in short). Assume that a pair $Q_i, i = 1, 2, ..., n$, is connected by a set S_i of paths and S_i contains $s_i \geq 1$ paths. Let $F = (F_1, F_2, ..., F_m)$ be the paths vector flow, where $m = \sum_{i=1}^{n} s_i$. Let the capacity restriction be

$$F \in C = \{F \in \mathbb{R}^m : 0 \leq \omega_p \leq F_p \leq \Omega_p, p = 1, 2, ..., m\},$$

where ω_p and Ω_p are given real numbers, and $C \subset \mathbb{R}^m$ is nonempty. Assume further that the travel cost on the path flow $F_p, p = 1, 2, ..., m$, depends on the whole path vector flow F and $T_p(F, \lambda) \geq 0$, where $\lambda \in \Lambda$ is a perturbing parametric. Then, the path cost vector is given by

$$T(F, \lambda) = (T_1(F, \lambda), T_2(F, \lambda), ..., T_m(F, \lambda)).$$

A path flow vector $\bar{F}$ is said to be an equilibrium flow if

$\forall Q_i, \forall \xi \in S_i, \forall \tau \in S_i$ such that

$$[T_\xi(\bar{F}, \lambda) < T_\tau(\bar{F}, \lambda)] \Rightarrow [\bar{F}_\xi = \Omega_\xi \ \text{ or } \ \bar{F}_\tau = \omega_\tau].$$

Suppose that the travel demands ψ_i of the O/D pair $Q_i, i = 1, 2, ..., n$, depend on $\lambda \in \Lambda$ and also on the equilibrium flows $\bar{F}$. Hence, considering all the O/D pairs, we have a

mapping $\psi : \mathbb{R}^m_+ \times \Lambda \to \mathbb{R}^n_+$. We use the Kronecker notation

$$\phi_{i\tau} = \begin{cases} 1 & \text{if } \tau \in S_i, \\ 0 & \text{if } \tau \notin S_i. \end{cases}$$

and

$$\phi = \{\phi_{i\tau}\}, \quad i = 1, 2, ..., n, \text{ and } \tau = 1, 2, ..., m.$$

Then, the path vector flows meetings the travel demands are called the feasible path vector flows and form the constraint set

$$K(\bar{F}, \lambda) = \{F \in C \mid \phi F = \psi(\bar{F}, \lambda)\}.$$

Lemma 3.2.12. *A path vector flow $\bar{F} \in K(\bar{F}, \lambda)$ is an equilibrium flow if and only if it is a solution of the following quasivariational inequality*
(TN) *finding $\bar{F} \in K(\bar{F}, \lambda)$ such that*

$$\langle T(\bar{F}, \lambda), H - \bar{F} \rangle \geq 0, \forall H \in K(\bar{F}, \lambda).$$

Let $X = W = \mathbb{R}^m, Z = \mathbb{R}^n, Y = \mathbb{R}, C_2(F^*) = \mathbb{R}_+, K_1(F, \lambda) = K_2(F, \lambda) = K(F, \lambda)$ and $A, Y, \Lambda, C_1(F)$, e_1 be as in problem (MBEP). Let $L(X, Y)$ be the space of all linear continuous operators from X into Y, and $T : Y \to L(X, P)$ be a vector function. We consider the following traffic network problems with equilibrium constraints.
(TNEC) finding $\bar{F}^* = (\bar{F}, \bar{\lambda}) \in \text{graph}S^{-1}$ such that

$$\langle T(\bar{F}^*), H^* - \bar{F}^* \rangle \geq 0, \forall H^* = (H, \lambda) \in \text{graph}S^{-1},$$

where $S(\lambda)$ is the solution set of (MSQEP). We denote the solution set of (TNEC) by Φ.

For $\varepsilon \in \mathbb{R}_+$, we denote the approximate solution set of (TNEC) by $\widetilde{\Phi}(\varepsilon)$.

$$\begin{aligned} \widetilde{\Phi}(\varepsilon) := \{F^* = (F, \lambda) \in \text{graph}S^{-1} \mid d(F, K(F, \lambda)) \leq \varepsilon, \\ f(F, H, \lambda) + \varepsilon e_1(F) \in C_1(F), \forall H \in K(F, \lambda), \\ \langle T(F^*), H^* - F^* \rangle + \varepsilon \geq 0, \ \forall H^* \in \text{graph}S^{-1}\}. \end{aligned}$$

Corollary 3.2.17. *Consider* (TNEC)*, assume that*

(i) *ψ is continuous*;

(ii) *f is C_1-upper semicontinuous*;

(iii) *the function $(F^*, H^*) \longmapsto \langle T(F^*), H^* - F^* \rangle$ is upper semicontinuous.*

Then, (TNEC) *is LP well-posed in the generalized sense if and only if* $\widetilde{\Phi}$ *is upper semi-continuous and compact-valued at* 0.

Corollary 3.2.18. *Suppose that all conditions in Corollary 3.2.17 are satisfied. Then,* (TNEC) *is LP well-posed if and only if*

$$\widetilde{\Phi}(\varepsilon) \neq \emptyset, \forall \varepsilon \geq 0, \ \ and \ \ \mathrm{diam}\widetilde{\Phi}(\varepsilon) \to 0 \ \ as \ \ \varepsilon \to 0.$$

Conclusions of Chapter 3

In this chapter, we obtained the following main results

- Establish the parametric bilevel vector equilibrium problems (MBEP). Afterwards, we study the semicontinuity, continuity of solution mappings for these problems (Theorem 3.1.1, Theorem 3.1.5, Theorem 3.1.8, Theorem 3.1.12 and Theorem 3.1.14). For the applications, we obtain the stability results for the parametric vector variational inequality problems with equilibrium constraints and parametric vector optimization problems with equilibrium constraints (Corollary 3.1.15 and Corollary 3.1.17).

- Establish the sufficient and necessary conditions of the Levitin-Polyak well-posedness for the reference problems and discuss some metric characterizations of these Levitin-Polyak well-posedness concepts in the behavior of approximate solution sets (Theorem 3.2.8, Theorem 3.2.9, Theorem 3.2.10 and Theorem 3.2.11). Application to traffic network problems with equilibrium constraints (Corollary 3.2.17 and Corollary 3.2.18)

These results were published in the article:

1. L. Q. Anh and N. V. Hung (2018), Stability of solution mappings for parametric bilevel vector equilibrium problems, *Computational and Applied Mathematics*, **37**, 1537–1549.

2. L. Q. Anh and N. V. Hung (2018), Levitin-Polyak well-posedness for strong bilevel vector equilibrium problems and applications to traffic network problems with equilibrium constraints, *Positivity*, **22**, 1223–1239.

GENERAL CONCLUSIONS AND RECOMMENDATIONS

1 General conclusions

In this thesis, we obtain some main results:

- Establish the parametric gap functions for (QEP_1) and (QEP_2) and two key hypotheses ($H_p(\gamma_0)$) and ($H_h(\gamma_0)$). Afterwards, we prove that these hypotheses are not only sufficient but also necessary for the Hausdorff lower semicontinuity and Hausdorff continuity of solution mappings to these problems. As an application, we derive several results on Hausdorff (lower) continuity properties of the solution mappings in the special cases of variational inequalities of the Minty type and the Stampacchia type.

- Establish gap function sequences for problems (WQEP) and $(\mathrm{WQEP})_n$ and Painlevé-Kuratowski upper convergence of solution sets for the reference problems. Base on the gap function sequences, we study the key hypotheses (H_h). Afterwards, we study necessary and sufficient conditions for Painlevé-Kuratowski lower convergence and Painlevé-Kuratowski convergence and application to vector quasivariational inequality.

- Establish the semicontinuity, continuity of solution mappings for the parametric bilevel vector equilibrium problems (MBEP) and application to parametric vector variational inequality problems with equilibrium constraints and parametric vector optimization problems with equilibrium constraints.

- Establish the sufficient and/or necessary conditions of the Levitin-Polyak well-posedness for bilevel vector equilibrium problems and discuss some metric characterizations of these Levitin-Polyak well-posedness concepts in the behavior of approximate solution sets and application to traffic network problems with equilibrium constraints.

2 Recommendations

Next time, we will continue the study on the following problems:

- Study the existence of solutions for optimization related problems as such as quasiequilibrium problems, quasivariational inequalities, bilevel equilibrium problems, variational inequality problems with equilibrium constraints,...

- Study the continuity property of solution sets for optimization related problems under weaker assumptions.

- Study several kinds of convergence for optimization related problems by using function and set sequences.

- Study some kinds of well-posed for optimization related problems by using gap functions.

LIST OF PUBLICATIONS RELATED TO THE THESIS

1. L. Q. Anh and N. V. Hung (2018), Gap functions and Hausdorff continuity of solution mappings to parametric strong vector quasiequilibrium problems, *Journal of Industrial and Management Optimization*, **14**, 65-79. (SCI-E)

2. L. Q. Anh and N. V. Hung (2018), Stability of solution mappings for parametric bilevel vector equilibrium problems, *Computational and Applied Mathematics*, **37**, 1537–1549. (SCI-E)

3. L. Q. Anh, T. Bantaojai, N. V. Hung, V. M. Tam and R. Wangkeeree (2018), Painlevé-Kuratowski convergences of the solution sets for generalized vector quasiequilibrium problems, *Computational and Applied Mathematics*, **37**, 3832–3845. (SCI-E)

4. L. Q. Anh and N. V. Hung (2018), Levitin-Polyak well-posedness for strong bilevel vector equilibrium problems and applications to traffic network problems with equilibrium constraints, *Positivity*, **22**, 1223–1239. (SCI-E)

THE RESULTS OF THE THESIS ARE REPORTED AT

1. New trends in Optimization Variational Analysis for Application, December 7-10, 2016.

2. The 9th Asian Conference on Fixed Point Theory and Optimizations, KMUTT, Bangkok, Thailand, May 19-22, 2016.

3. The 14th Work shop on Optimization and Scientific Computing Hanoi, April 21-23, 2016

4. Seminars of the Analysis section of Institute of Natural Sciences Education, Vinh University;

5. The 15th Work shop on Optimization and Scientific Computing Hanoi, April 20-22, 2017.

6. The 9th Vietnam Mathematical Congress, Nha Trang, August 14-18, 2018.