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DUONG XUAN GIAP

THE ERGODIC THEOREMS AND THE STRONG LAW OF LARGE NUMBERS FOR ARRAY OF RANDOM SETS

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Supervisor: 1. Prof. Dr. Nguyen Van Quang

2. Prof. Charles Castaing

Reviewer 1: Prof. Dr. Sc. Dang Hung Thang

Hanoi University of Science, VNU

Reviewer 2: Assoc. Prof. Dr. Tran Hung Thao

Institute of Mathematics - Vietnam Academy of Science and Technology

Reviewer 3: Dr. Le Hong Son

Vinh University of Technology Education

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PREFACE

1. Rationale

Recently, the ergodic theorems and the strong laws of large numbers for set-valued random variables have been studied and applied in stochastic optimization, statistics, mathematical economics, medicine and some other areas. It is well known that the theory of set-valued random variables is a natural extension of that of general random elements. Hence, the researching of ergodic theorems and the laws of large numbers for random sets has not only theoretical meanings, but also practical meanings.

The practice requires us to study the multi-indexed array of random variables. For multi-indexed structure, the usual partial order relation in the index set $\mathbb{N}^d$ is not complete. So, if we extend the limit theorems for random sets from the sequence case to the multi-indexed array as $\mathbf{n}_{\max} \to \infty$ or as $\mathbf{n}_{\min} \to \infty$, then we will have a lot of new things. This makes the results of multi-valued strong laws of large numbers and ergodic theorems more interesting.

The study of ergodic theorems was started in $1931 - 1932$ by J. v. Neumann and G. D. Birkhoff, having its origins in statistical mechanics. In recent decades, the Birkhoff's ergodic theorem (BET) has been considerable interest in two directions: to multidimensional structure and to random sets. In the first direction, in 1951, N. Dunford and A. Zygmund established the BET for noncommutative families of measure-preserving transformations with discrete and continuous parameters, respectively. Then later, N. Dunford, J. T. Schwartz (in 1956) and N. A. Fava (in 1972) generalized these results to those at the operator theoretic level. Moreover, many authors generalized them in the direction of weighted averages such as R. L. Jones and J. Olsen (in 1994), M. Lin and M. Weber (2007), F. Mukhamedov, M. Mukhamedov and S. Temir (in 2008), etc. In the second direction, in 1991, J. Bán established the BET for compact valued random sets or fuzzy random sets in a Banach space with respect to the convergence in the Hausdorff distance. In 2003, C. Choirat, C. Hess and R. A. Seri obtained the BET for convex valued random sets with respect to the Kuratowski convergence. Recently, in 2011, H. Ziat proved the BET for random sets with the types of convergence: Mosco, Wijsman and Slice. Hence, the studying BET for both multidimensional structure and random sets is an up-to-date tendency.

The first multivalued strong law of large numbers was proved by Z. Artstein and R. A. Vitale in 1975 for independent identically distributed (i.i.d) random variables whose values are compact subsets of $\mathbb{R}^d$, with the convergence in Hausdorff distance. This result

is extended in two directions: to compact valued random variables and to closed valued random variables. In the first direction, we can refer to some works of the following authors: N. Cressie (in 1978), C. Hess (in 1979), M. L. Puri and D. A. Ralescu (in 1983), F. Hiai (in 1984), Z. Artstein and J. C. Hansen (in 1985), P. Terán and I. Molchanov (in 2006), etc. In the second direction, the strong law of large numbers was proved at first in 1981 by Z. Artstein and S. Hart on Kuratowski for i.i.d. random variables taking values are closed subsets of $\mathbb{R}^d$. Later, it was studied by F. Hiai and C. Hess for Mosco convergence and Wijsman convergence. Hence, the studying strong law of large numbers for random sets is also an up-to-date tendency of the probability theory.

Most of the results for strong laws of large numbers was concerned with i.i.d random variables. While it is not always possible to assume that the random variables are independent, they can be often dependent. A new direction for multi-valued strong law of large numbers is studying for the following kinds of dependency of random sets: pair independent, exchangeable, 2-exchangeable. This direction has practical meanings.

The strong law of large numbers and the ergodic theorems are usually studied to the random variables taking values are compact or convex or closed subsets, ... of a Banach space. So, the results in this direction and their proofs are the combination between probability theory, convex analysis and functional analysis.

The convergence in the Hausdorff distance is usually used to the compact valued random variables. For the closed valued random variables, one usually use the types of convergence: Kuratowski, Mosco and Wijsman. The Kuratowski convergence is well-suited to random sets in the finite-dimensional space. The Mosco convergence is an extension of Kuratowski convergence to the case of random sets in the Banach space. This type of convergence is well-suited to the reflexive Banach spaces and are most helpful for applications to the variational inequalities. For non reflexive Banach spaces, the Wijsman convergence has been introduced and is well-suited to study the rates of convergence, and is able to allow us to prove the strong law of large numbers with respect to Slice convergence which is most helpful for applications to stochastic optimization. Thus, the studying limit theorems for random sets with respect to the Mosco convergence and the Wijsman convergence has a lot of meanings.

With the above reasons, we have chosen the topic for the thesis that is: **"The ergodic theorems and the strong laws of large numbers for array of random sets"**.

2. Objective of the research

The objective of the thesis is to establish the multidimensional Birkhoff's ergodic

theorem, to establish the strong laws of large numbers for double array and triangular array of random sets with several conditions.

3. Subject of the research

- The multidimensional Birkhoff's ergodic theorem.
- The strong laws of large numbers for array of random sets.

4. Scope of the research

The multidimensional Birkhoff's ergodic theorem, the strong laws of large numbers for double array and for triangular array of random sets in a separable real Banach space. The types of convergence are considered: Mosco convergence and Wijsman convergence. For the multi-valued strong law of large numbers, the random sets are required: independent, or pair independent, or 2-exchangeable.

5. Methodology of the research

We use theoretical research method belonging to the areas: *probability theory*, *convex analysis* and *functional analysis*. Technically, we use convexification technique, types of Stolz's theorem, etc.

6. Contribution of the thesis

The results of thesis contribute more abundant for the researching directions of limit theorems in probability theory.

The thesis is contribution of material for the students, the master students, the doctoral students belonging to the speciality: Theory of Probability and Mathematical Statistics.

7. Organization of the research

7.1. Overview of the research

In this thesis, we establish the BET and the strong law of large numbers for array of random sets with respect to Mosco convergence and Wijsman convergence.

At first, we introduce some basic notions of probability theory on the space of closed subsets of Banach space. Later, we prove some results of convergence relatively to the topologies Mosco and Wijsman for multi-indexed array of closed subsets of separable real Banach space and for multi-indexed array of random sets.

For the ergodic theorems, we establish the multidimensional BET for the cases: single-values and multi-values. Especially, the mult-valued BET is proved for two-indexed structure.

For the strong law of large numbers for double array of random sets, we consider for the case as $m \vee n \to \infty$. Combine the type of Stolz's theorem for double array, the property of convergence as $m \vee n \to \infty$, the convexification technique for double array and the previous lemmas, we establish the strong law of large numbers on Mosco convergence and Wijsman convergence for double array of random sets. The random sets are required: p.i.i.d, or independent and taking closed-values of Rademacher type p Banach space, or 2-exchangeable.

For the strong law of large numbers for triangular array of random sets, we establish the strong law of large numbers on Mosco convergence and Wijsman convergence for rowwise independent random sets taking closed-values of Rademacher type p Banach space. To do this, we construct type of Stolz's theorem for the case of triangular array.

To establish the BET and the strong law of large numbers with respect to Mosco convergence and Wijsman convergence, we develop the convexification technique from the sequence case to double array case and to triangular array case.

7.2. The organization of the research

Besides the sections of usual notations, preface, general conclusions and recommendations, list of the author's articles related to the thesis and references, the thesis is organized into four chapters.

Chapter 1 presents the basic knowledge of the family of all closed subsets of a Banach space, the properties of functional analysis and convex analysis, establish some results of convergence relatively to the topologies Mosco and Wijsman for array of closed subsets of a Banach space and for array of random sets. Section 1.1 presents the preliminaries consist of the notations, the definitions and the basic notions relatively to main results of thesis. Section 1.2 presents the definition of the types of usual convergence in hyperspace of a Banach space and prove some results of convergence relatively to the topologies Mosco and Wijsman for multi-indexed array. Section 1.3 presents the results of convergence relatively to the topologies Mosco and Wijsman for multi-indexed array of random sets. These results are used to prove the BET and the multi-valued strong law of large numbers in the next chapters.

Chapter 2 presents the BET for the multidimensional structure of single-valued or multi-valued random variables. Section 2.1 presents some basic concepts and basic properties of ergodic theory used to the next sections. In Section 2.2, we establish the multidimensional BET for random elements in a separable real Banach space. This is a key tool to establish the multi-valued BET having multidimensional structure. Section 2.3 studies the two-dimensional BET for random sets with respect to Mosco convergence and

Wijsman convergence. In this section, we also prove the multidimensional multi-valued BET without the ergodicity of the measure-preserving transformations. Section 2.4 studies the two-dimensional BET for fuzzy random sets with respect to Mosco convergence.

Chapter 3 studies the strong law of large numbers for double array of random sets with respect to Mosco convergence and Wijsman convergence. Section presents the essential lemmas to prove main results of Chapter 3. Section 3.2 studies the strong law of large numbers for double array of random sets with the cases: p.i.i.d, or independent and taking closed-values of Rademacher type p Banach space, or 2-exchangeable.

Chapter 4 introduce the strong law of large numbers for triangular array of random sets with respect to Mosco convergence and Wijsman convergence. Section 4.1 studies the Stolz's theorem for triangular array case. Section 4.2 studies the strong law of large numbers for triangular array of rowwise independent random sets taking closed-values of Rademacher type p Banach space.

CHAPTER 1

SOME PROPERTIES OF MOSCO CONVERGENCE AND WIJSMAN CONVERGENCE

In this chapter, we introduce some basic notions of probability theory on the space of closed subsets of Banach space, study the types of convergence and the important properties of functional analysis and convex analysis in this space. We establish some results of convergence relatively to the topologies Mosco and Wijsman for multi-indexed array of closed subsets of separable real Banach space and for multi-indexed array of random sets. Main results of this chapter are based on the paper [1].

1.1. Preliminaries

In this thesis, if not added assume, we suppose that $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space, $\mathcal{F}$ is an sub-σ-field of $\mathcal{A}$, $(\mathfrak{X}, \|\cdot\|)$ is a separable real Banach space, $\mathcal{B}_{\mathfrak{X}}$ is the Borel σ-field of $\mathfrak{X}$, $\mathfrak{X}^*$ is the dual space of $\mathfrak{X}$. Let $c(\mathfrak{X})$ be the family of all nonempty closed subsets of $\mathfrak{X}$.

We denoted by $\mathbb{N}$ (resp. $\mathbb{Q}$) (resp. $\mathbb{R}$) (resp. $\mathbb{R}_+$) the set of all positive integers (resp. rational numbers) (resp. real numbers) (resp. nonnegative real numbers).

For each $d \in \mathbb{N}$, on the set $\mathbb{N}^d$, the members $(1, 1, \ldots, 1)$, $(2, 2, \ldots, 2)$, $(m_1, m_2, \ldots, m_d)$, $(n_1, n_2, \ldots, n_d)$ are respectively denoted by $\mathbf{1}$, $\mathbf{2}$, $\mathbf{m}$, $\mathbf{n}$. Let $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{N}^d$, we set $|\mathbf{n}| = \prod_{i=1}^{d} n_i$, $\mathbf{n}_{\max} = \max\{n_i : i = 1, 2, \ldots, d\}$ and $\mathbf{n}_{\min} = \min\{n_i : i = 1, 2, \ldots, d\}$. For two real numbers m and n, we denote $\max\{m, n\}$ (resp. $\min\{m, n\}$) by $m \vee n$ (resp. $m \wedge n$). For each $a \in \mathbb{R}$, the logarithm to the base 2 of $a \vee 1$ will be denoted by $\log^+ a$. For $\mathbf{m}, \mathbf{n} \in \mathbb{N}^d$, we write $\mathbf{m} \preceq \mathbf{n}$ (resp. $\mathbf{m} \prec \mathbf{n}$) if $m_i \leqslant n_i$ (resp. $m_i < n_i$) for every $i = 1, 2, ..., d$.

For $A, B \subset \mathfrak{X}$, $\mathrm{cl}A$ and $\overline{\mathrm{co}}A$ denote the *norm-closure* and the *closed convex hull* of A, respectively; the *distance function* $d(\cdot, A)$ of A, the *Hausdorff distance* $d_{\mathrm{H}}(A, B)$ between two sets A and B, the *support function* $s(\cdot, A)$ of A, the *norm* $\|A\|$ of A are defined by

$$d(x, A) = \inf\{\|x - y\| \ : y \in A\}, (x \in \mathfrak{X}),$$
$$d_{\mathrm{H}}(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\},$$
$$s(x^*, A) = \sup\{\langle x^*, y\rangle \ : y \in A\}, (x^* \in \mathfrak{X}^*),$$
$$\|A\| = \sup\{||x|| \ : x \in A\}.$$

Put $\mathbf{B}^* = \{x^* \in \mathfrak{X}^* : \|x^*\| \leq 1\}$ and $\mathbf{S}^* = \{x^* \in \mathfrak{X}^* : \|x^*\| = 1\}$. Then, $\mathbf{B}^*$ (resp. $\mathbf{S}^*$) is called the *closed unit ball* (resp. the *unit sphere*) of $\mathfrak{X}^*$.

Let $\mathcal{P}(\mathfrak{X})$ be the family of all nonempty subsets of $\mathfrak{X}$. In $\mathcal{P}(\mathfrak{X})$, one defined Minkowski addition and scalar multiplication as follows:

$$A + B = \{a + b : a \in A, b \in B\},$$
$$\lambda A = \{\lambda a : a \in A\},$$

where $A, B \in \mathcal{P}(\mathfrak{X})$, $\lambda \in \mathbb{R}$. $\mathcal{P}(\mathfrak{X})$ is not a linear space with respect to the above addition and multiplication, since we can not find any inverse of a set A in general.

Let $\mathcal{B}_{c(\mathfrak{X})}$ be the σ-field on $c(\mathfrak{X})$ generated by the sets $U^- := \{C \in c(\mathfrak{X}) : C \cap U \neq \emptyset\}$ taken for all open subsets U of $\mathfrak{X}$. We call $\mathcal{B}_{c(\mathfrak{X})}$ the *Effrös σ-field*.

1.1.1 Definition. A mapping $F : \Omega \to c(\mathfrak{X})$ is said to be *$\mathcal{F}$-measurable* if for every $B \in \mathcal{B}_{c(\mathfrak{X})}$, $F^{-1}(B) \in \mathcal{F}$. The mapping $\mathcal{F}$-measurable F is also called *$\mathcal{F}$-measurable random set*. If $\mathcal{F} = \mathcal{A}$ then F is said for shortly to be *random set*.

The operations to random sets are defined as ones in $\mathcal{P}(\mathfrak{X})$ for each $\omega \in \Omega$.

Given the random set F, we define a sub-σ-field $\mathcal{A}_F$ of $\mathcal{A}$ by $\mathcal{A}_F = \{F^{-1}(B) : B \in \mathcal{B}_{c(\mathfrak{X})}\}$. Then, $\mathcal{A}_F$ is the smallest sub-σ-field of $\mathcal{A}$ with respect to which F is measurable. The *distribution* of F is a probability measure $\mathbb{P}_F$ on $\mathcal{B}_{c(\mathfrak{X})}$ defined by $\mathbb{P}_F(B) = \mathbb{P}(F^{-1}(B)), B \in \mathcal{B}_{c(\mathfrak{X})}$.

1.1.3 Definition. A family of random sets $\{F_i : i \in I\}$ is said to be *independent* (resp. *pair independent*) if $\mathcal{A}_{F_i}, i \in I$, are independent (resp. pair independent), and is said to be *identically distributed* if all $\mathbb{P}_{F_i}$, $i \in I$, are identical.

1.1.4 Definition. A finite family of random sets $\{F_1, F_2, \ldots, F_n\}$ is said to be *exchangeable* if for every permutation π of $\{1, 2, \ldots, n\}$ and $\{B_1, B_2, \ldots, B_n\}$ of $\mathcal{B}_{c(\mathfrak{X})}$,

$$\mathbb{P}(F_1 \in B_1, \ldots, F_n \in B_n) = \mathbb{P}(F_{\pi(1)} \in B_1, \ldots, F_{\pi(n)} \in B_n).$$

A countable family of random sets is said to be *exchangeable* if all its finite subsets are exchangeable.

1.1.5 Definition. A family of random sets $\{F_i : i \in I\}$ is called 2-*exchangeable* if for every $i_1, i_2, j_1, j_2 \in I$, $i_1 \neq i_2$, $j_1 \neq j_2$ and $B_1, B_2 \in \mathcal{B}_{c(\mathfrak{X})}$,

$$\mathbb{P}(F_{i_1} \in B_1, F_{i_2} \in B_2) = \mathbb{P}(F_{j_1} \in B_1, F_{j_2} \in B_2).$$

The relations between the independent identically distribution (i.i.d), the pair independent identically distribution (p.i.i.d), the exchangeability, the 2-exchangeability and the identically distribution of random sets, are described as follows:

$$\begin{tikzcd}
\text{i.i.d} \arrow[r] \arrow[d] & \text{p.i.i.d} \arrow[d] \\
\text{the exchangeability} \arrow[r] & \text{the 2-exchangeability} \arrow[d] \\
 & \text{the identically distribution}
\end{tikzcd}$$

For $p \geq 1$, $\mathbf{L}^p(\mathcal{F}, \mathfrak{X})$ denoted the Banach space of $\mathcal{F}$-measurable random elements $f : \Omega \to \mathfrak{X}$ such that $\| f \|_p = (\mathbb{E} \| f \|^p)^{\frac{1}{p}} < \infty$. If $\mathcal{F} = \mathcal{A}$ then $\mathbf{L}^p(\mathcal{A}, \mathfrak{X})$ is denoted for shortly by $\mathbf{L}^p(\mathfrak{X})$. If $\mathfrak{X} = \mathbb{R}$ then $\mathbf{L}^p(\mathbb{R})$ is replaced by $\mathbf{L}^p$.

For $p \geq 1$ and for each random set $\mathcal{F}$-measurable F, we set

$$\mathcal{S}_F^p(\mathcal{F}) = \{f \in \mathbf{L}^p(\mathcal{F}, \mathfrak{X}) : f(\omega) \in F(\omega) \text{ a.s.}\}$$

In the case $\mathcal{F} = \mathcal{A}$, $\mathcal{S}_F^p(\mathcal{A})$ is denoted for shortly by $\mathcal{S}_F^p$.

1.1.8 Definition. The random set $F : \Omega \to c(\mathfrak{X})$ is called *integrable* if $\mathcal{S}_F^1$ is nonempty.

In 1965, R. J. Aumann introduced the concept of the expectation of random set as follows.

1.1.9 Definition. The *expectation* $\mathbb{E}F$ of integrable random set F is defined by

$$\mathbb{E}F := \{\mathbb{E}f : f \in \mathcal{S}_F^1\},$$

where $\mathbb{E}f$ is the usual Bochner integral of the random element f.

Moreover, for each random set $\mathcal{F}$-measurable F, we put

$$\mathbb{E}(F, \mathcal{F}) := \{\mathbb{E}f : f \in \mathcal{S}_F^1(\mathcal{F})\}.$$

1.1.10 Definition. Suppose that $\{r_j : j \geqslant 1\}$ is a sequence of i.i.d random variables and $\mathbb{P}(r_1 = 1) = \mathbb{P}(r_1 = -1) = \frac{1}{2}$. The Banach space $\mathfrak{X}$ is said to be *Rademacher type p*

($p \in [1,2]$) if there exists a constant $C > 0$ such that for every $i \geqslant 1$ and every $v_j \in \mathfrak{X}$ ($1 \leqslant j \leqslant i$),

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^{i} r_j v_j\Big\|^p\Big)^{1/p} \leqslant C\Big(\sum_{j=1}^{i} \|v_j\|^p\Big)^{1/p}.$$

Let $\{x_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\} \subset \mathbb{R}$, we put

$$\liminf_{\mathbf{n}_{\max}\to\infty} x_{\mathbf{n}} = \sup_{k\geq 1} \inf_{\mathbf{n}_{\max}\geq k} x_{\mathbf{n}}, \qquad \limsup_{\mathbf{n}_{\max}\to\infty} x_{\mathbf{n}} = \inf_{k\geq 1} \sup_{\mathbf{n}_{\max}\geq k} x_{\mathbf{n}},$$
$$\liminf_{\mathbf{n}_{\min}\to\infty} x_{\mathbf{n}} = \sup_{k\geq 1} \inf_{\mathbf{n}_{\min}\geq k} x_{\mathbf{n}}, \qquad \limsup_{\mathbf{n}_{\min}\to\infty} x_{\mathbf{n}} = \inf_{k\geq 1} \sup_{\mathbf{n}_{\min}\geq k} x_{\mathbf{n}}.$$

Let s (resp. w) be the *strong* (resp. *weak*) topology on $\mathfrak{X}$.

1.1.11 Definition.

(a) An array $\{x_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\} \subset \mathbb{R}$ *converges to* $x \in \mathbb{R}$ as $\mathbf{n}_{\max} \to \infty$ if $\liminf_{\mathbf{n}_{\max}\to\infty} x_{\mathbf{n}} = \limsup_{\mathbf{n}_{\max}\to\infty} x_{\mathbf{n}} = x$. In this case, we shall write $\lim_{\mathbf{n}_{\max}\to\infty} x_{\mathbf{n}} = x$, or $x_{\mathbf{n}} \to x$ as $\mathbf{n}_{\max} \to \infty$.

(b) An array $\{x_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\} \subset \mathfrak{X}$ *converges to* $x \in \mathfrak{X}$ as $\mathbf{n}_{\max} \to \infty$ if $\lim_{\mathbf{n}_{\max}\to\infty} \|x_{\mathbf{n}} - x\| = 0$. In this case, we write s-$\lim_{\mathbf{n}_{\max}\to\infty} x_{\mathbf{n}} = x$, or $x_{\mathbf{n}} \overset{s}{\to} x$ as $\mathbf{n}_{\max} \to \infty$ (for shortly, we usually remove the notation s).

(c) An array $\{x_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\} \subset \mathfrak{X}$ *weakly converges to* $x \in \mathfrak{X}$ as $\mathbf{n}_{\max} \to \infty$ if $\lim_{\mathbf{n}_{\max}\to\infty} \langle x^*, x_{\mathbf{n}}\rangle = \langle x^*, x\rangle$ for every $x^* \in \mathfrak{X}^*$. In this case, we write w-$\lim_{\mathbf{n}_{\max}\to\infty} x_{\mathbf{n}} = x$, or $x_{\mathbf{n}} \overset{w}{\to} x$ as $\mathbf{n}_{\max} \to \infty$.

The convergence as $\mathbf{n}_{\min} \to \infty$ is stated in the similar way.

1.1.15 Definition. The array of random elements $\{f_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\}$ *converges in the* r^{th} *mean* ($r > 0$) to a random element f as $\mathbf{n}_{\max} \to \infty$ (resp. $\mathbf{n}_{\min} \to \infty$) denoted by $f_{\mathbf{n}} \to f$ in $\mathbf{L}^r$ as $\mathbf{n}_{\max} \to \infty$ (resp. $\mathbf{n}_{\min} \to \infty$), if $\mathbb{E}\|f_{\mathbf{n}} - f\|^r \to 0$ as $\mathbf{n}_{\max} \to \infty$ (resp. $\mathbf{n}_{\min} \to \infty$).

1.2. Some properties of Mosco convergence and Wijsman convergence for array of closed subsets of a Banach space

At first, we introduce some important types of convergence on the space of closed subsets of Banach space. Let $d \in \mathbb{N}$ and let $\{A_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\}$ be an array in $c(\mathfrak{X})$. For notational convenience, the topologies s and w on $\mathfrak{X}$ are denoted by t. We put

$$t\text{-}\liminf_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} = \{x \in \mathfrak{X} : x = t\text{-}\lim_{\mathbf{n}_{\max}\to\infty} x_{\mathbf{n}},\ \text{for } x_{\mathbf{n}} \in A_{\mathbf{n}}\},$$

$$t\text{-}\limsup_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} = \{x \in \mathfrak{X} : x = t\text{-}\lim_{\mathbf{k}_{\max}\to\infty} x_{\mathbf{k}},\ \text{for } x_{\mathbf{k}} \in A_{\mathbf{n}(\mathbf{k})}\},$$

where $\{A_{\mathbf{n}(\mathbf{k})} : \mathbf{k} \in \mathbb{N}^d\}$ is a sub-array of $\{A_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\}$ (the *sub-array* is subsequence with respect to each coordinate).

It is not hard to see that $t\text{-}\liminf_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} \subset t\text{-}\limsup_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}}$ and $s\text{-}\liminf_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} \subset w\text{-}\limsup_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}}$.

1.2.1 Definition. Let $A \in c(\mathfrak{X})$. The array $\{A_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\} \subset c(\mathfrak{X})$ is said to be

(*a*) *converges in the Hausdorff distance* to A as $\mathbf{n}_{\max} \to \infty$ and is denoted by $\mathrm{H}\text{-}\lim_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} = A$, if $\lim_{\mathbf{n}_{\max}\to\infty} d_{\mathrm{H}}(A_{\mathbf{n}}, A) = 0$;

(*b*) *weak converges* to A as $\mathbf{n}_{\max} \to \infty$ and is denoted by $\mathrm{W}\text{-}\lim_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} = A$, if $\lim_{\mathbf{n}_{\max}\to\infty} s(x^*, A_{\mathbf{n}}) = s(x^*, A)$ for every $x^* \in \mathfrak{X}^*$;

(*c*) *converges in the sense of Wijsman* to A as $\mathbf{n}_{\max} \to \infty$ and is denoted by $\mathrm{Wijs}\text{-}\lim_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} = A$, if $\lim_{\mathbf{n}_{\max}\to\infty} d(x, A_{\mathbf{n}}) = d(x, A)$ for every $x \in \mathfrak{X}$;

(*d*) *converges in the sense of Kuratowski* to A with respect to t as $\mathbf{n}_{\max} \to \infty$ and is denoted by $t\text{-}\lim_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} = A$, if $t\text{-}\limsup_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} = t\text{-}\liminf_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} = A$;

(*e*) *converges in the sense of Mosco* to A as $\mathbf{n}_{\max} \to \infty$ and is denoted by $\mathrm{M}\text{-}\lim_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} = A$, if

$$w\text{-}\limsup_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} = s\text{-}\liminf_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} = A.$$

The convergence as $\mathbf{n}_{\min} \to \infty$ is stated in the similar way.

We give the following property to prove the main results.

1.2.3 Theorem. *Let $\{A_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\} \subset c(\mathfrak{X})$. Then, if $s\text{-}\liminf_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} \neq \emptyset$ then $s\text{-}\liminf_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} \in c(\mathfrak{X})$.*

The following theorem is established to prove the "lim sup" part of Mosco convergence of strong law of large numbers for array of random sets.

1.2.5 Theorem. *Let $\{A, A_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\} \subset c(\mathfrak{X})$ and D^* be a countable subset of $\mathbf{S}^*$ such that $x \in \overline{\mathrm{co}}A$ if and only if $\langle x^*, x\rangle \leq s(x^*, A)$ for every $x^* \in D^*$. Then, if $\limsup_{\mathbf{n}_{\max}\to\infty} s(x^*, A_{\mathbf{n}}) \leq s(x^*, A)$ for every $x^* \in D^*$, then*

$$w\text{-}\limsup_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} \subset \overline{\mathrm{co}}A.$$

The following result is used to prove the "lim inf" part of Wijsman convergence for multi-indexed structure.

1.2.7 Theorem. *Suppose that $\{A, A_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\} \subset c(\mathfrak{X})$ and D^* is a countable dense subset of $\mathbf{B}^*$ such that $d(x, \overline{\mathrm{co}}A) = \sup_{x^*\in D^*} \{\langle x^*, x\rangle - s(x^*, \overline{\mathrm{co}}A)\}$, for every $x \in \mathfrak{X}$. If for*

every $x^* \in D^*$, $\limsup_{\mathbf{n}_{\max}\to\infty} s(x^*, A_{\mathbf{n}}) \le s(x^*, A)$ *then for every* $x \in \mathfrak{X}$

$$\liminf_{\mathbf{n}_{\max}\to\infty} d(x, A_{\mathbf{n}}) \ge d(x, \overline{\mathrm{co}}A).$$

Studying the relation between the Wijsman convergence and the Kuratowski convergence for the case of multi-indexed array, we obtain the following result.

1.2.8 Theorem. *Let* $\{A, A_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\} \subset c(\mathfrak{X})$. *If* Wijs-$\lim_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} = A$ *then* s-$\lim_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} = A$.

1.3. Some properties of Mosco convergence and Wijsman convergence for array of random sets

In Definition 1.2.1, if we replace $A_{\mathbf{n}}$ by $F_{\mathbf{n}}(\omega)$ and replace A by $F(\omega)$ for ω in a set with probability 1, where F, $F_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{N}^d$, are random sets, then we obtain the definition of almost sure convergence for random sets.

Based on the results in Section 1.2, we obtain two following theorem of the "lim sup" part of Wijsman convergence for the multi-indexed array of random sets.

1.3.2 Theorem. *Suppose that* D *is a countable dense subset of* $\mathfrak{X}$ *and* $F, F_{\mathbf{n}}$ $(\mathbf{n} \in \mathbb{N}^d)$ *are random sets. If for each* $x \in D$, $\limsup_{\mathbf{n}_{\max}\to\infty} d(x, F_{\mathbf{n}}(\omega)) \le d(x, F(\omega))$ a.s., *then*

$$\limsup_{\mathbf{n}_{\max}\to\infty} d(x, F_{\mathbf{n}}(\omega)) \le d(x, F(\omega)) \text{ for every } x \in \mathfrak{X} \text{ a.s.}$$

1.3.3 Theorem. *Let* $F, F_{\mathbf{n}}$ $(\mathbf{n} \in \mathbb{N}^d)$ *be random sets. If* $F(\omega) \subset s$-$\liminf_{\mathbf{n}_{\max}\to\infty} F_{\mathbf{n}}(\omega)$ *a.s., then*

$$\limsup_{\mathbf{n}_{\max}\to\infty} d(x, F_{\mathbf{n}}(\omega)) \le d(x, F(\omega)) \text{ for every } x \in \mathfrak{X} \text{ a.s.}$$

The following is a property of Wijsman convergence for multi-indexed array of random sets.

1.3.4 Theorem. *Suppose that* D *is a countable dense subset of* $\mathfrak{X}$ *and* $F, F_{\mathbf{n}}$ $(\mathbf{n} \in \mathbb{N}^d)$ *are random sets. Then, the array* $\{F_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\}$ *converges in the sense of Wijsman to* F *a.s. as* $\mathbf{n}_{\max} \to \infty$ *if and only if for each* $x \in D$,

$$d(x, F_{\mathbf{n}}(\omega)) \to d(x, F(\omega)) \text{ a.s. as } \mathbf{n}_{\max} \to \infty.$$

1.4 Remark. The results in this chapter are considered to the case of convergence as $\mathbf{n}_{\max} \to \infty$. For the case of convergence as $\mathbf{n}_{\min} \to \infty$, we obtain the similar results.

The conclusions of Chapter 1

In this chapter, we obtain some main results:

- Give some properties of Mosco convergence and Wijsman convergence for multi-indexed array of closed subsets of separable real Banach space.

- Establish some results of convergence for multi-indexed array of random sets with respect to Mosco convergence and Wijsman convergence.

CHAPTER 2

THE MULTIDIMENSIONAL BIRKHOFF'S ERGODIC THEOREM

In this chapter, we introduce some notions relatively to the ergodic theory, establish the multidimensional Birkhoff's ergodic theorem in separable real Banach and obtain two-dimensional multi-valued Birkhoff's ergodic theorem for random sets and for fuzzy random sets. Main results of this chapter are based on the paper [3].

2.1. Preliminaries

2.1.1 Definition. (i) A transformation $T : \Omega \to \Omega$ is said to be *measurable* if $T^{-1}(A) \in \mathcal{A}$ for every $A \in \mathcal{A}$.

(ii) A transformation $T : \Omega \to \Omega$ is said to be *measure-preserving* if T is measurable and $\mathbb{P}(T^{-1}(A)) = \mathbb{P}(A)$ for every $A \in \mathcal{A}$. In this case, we also say that $\mathbb{P}$ is T-*invariant measure*.

(iii) A set $A \in \mathcal{A}$ is said to be T-*invariant* if $T^{-1}(A) = A$.

(iv) A random variable f is said to be T-*invariant* if $f \circ T = f$.

(v) A measure-preserving transformation $T : \Omega \to \Omega$ is said to be *ergodic* if for every $A \in \mathcal{A}$, the condition $T^{-1}(A) = A$ yields $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.

2.1.2 Remark. The family of all T-invariant subset of $\mathcal{A}$ constitute a sub-σ-field of $\mathcal{A}$. This σ-field is denoted by $\mathcal{I}_T$.

If $T_1, T_2 : \Omega \to \Omega$ are measure-preserving transformations then $T_1 \circ T_2$ (which is also written for shortly by T_1T_2) is a measure-preserving transformation, too. Especially, if $T : \Omega \to \Omega$ is a measure-preserving transformation then T^n $(n \in \mathbb{N})$ is a measure-preserving transformation, too.

Next, we introduce some basic concepts of fuzzy random sets. It is an extension of random set.

A mapping $u : \mathfrak{X} \to [0, 1]$ is called a *fuzzy set* in $\mathfrak{X}$.

For each fuzzy set u, the α-*level set* $L_\alpha u$ $(\alpha \in (0, 1])$ is denoted by

$$L_\alpha u = \{x \in \mathfrak{X} : u(x) \geq \alpha\}.$$

We also define $L_{\alpha^+}u = \{x \in \mathfrak{X} : u(x) > \alpha\}$, $\alpha \in [0,1)$.

Let $\mathbf{F}(\mathfrak{X})$ denote the space of fuzzy sets $u : \mathfrak{X} \to [0,1]$ such that

(1) u is normal, i.e., the 1-level set $L_1 u$ is nonempty,

(2) u is upper semicontinuous, that is, for each $\alpha \in (0,1]$, the α-level set $L_\alpha u$ is a closed subset of $\mathfrak{X}$.

In $\mathbf{F}(\mathfrak{X})$, we define the following operations:

$$(u+v)(x) = \sup_{y+z=x} \min\{u(y), v(z)\},$$
$$(\lambda u)(x) = \begin{cases} u(\lambda^{-1}x) & \text{if } \lambda \neq 0, \\ \mathbb{I}_{\{0\}}(x) & \text{if } \lambda = 0, \end{cases}$$

where $u, v \in \mathbf{F}(\mathfrak{X})$, $\lambda \in \mathbb{R}$.

The *closed convex hull* $\overline{\text{co}}u$ of $u \in \mathbf{F}(\mathfrak{X})$ is defined as follows:

$$\overline{\text{co}}u(x) = \sup\{\alpha \in (0,1] : x \in \overline{\text{co}}(L_\alpha u)\}.$$

2.1.3 Definition. A mapping $\tilde{F} : \Omega \to \mathbf{F}(\mathfrak{X})$ is said to be *fuzzy random set* if $\{(\omega, x) : x \in L_\alpha(\tilde{F}(\omega))\} \in \mathcal{A} \times \mathcal{B}_{\mathfrak{X}}$, for every $\alpha \in (0,1]$.

In 1991, J. Bán showed that $\tilde{F}$ is the fuzzy random set then $L_\alpha \tilde{F}$ is a random set for every $\alpha \in (0,1]$.

2.1.4 Definition. The *expected value* of fuzzy random set $\tilde{F}$, denoted by $\mathbb{E}\tilde{F}$, is a fuzzy set in $\mathfrak{X}$ such that $L_\alpha\left(\mathbb{E}\tilde{F}\right) = \mathbb{E}\left(L_\alpha \tilde{F}\right)$ for every $\alpha \in (0,1]$.

2.2. The multidimensional Birkhoff's ergodic theorem for random variables taking values in separable real Banach space

In 1951, N. Dunford established the multidimensional real-valued Birkhoff's ergodic theorem, where the limit function is a random variable. This result was extended by N. Dunford, J. T. Schwartz (in 1956) and by N. A. Fava (in 1972) to the contractions. In following, we establish the multidimensional Birkhoff's ergodic theorem for the case of separable real Banach space-valued random variables. This result show that the limit function is the conditional expectation with respect to the σ-field of invariant sets.

2.2.2 Theorem. *Suppose that $T_1, T_2, \ldots, T_d$ are the commutative measure-preserving transformations. If the random element f satisfying $\mathbb{E}\left(\|f\|\left(\log^+\|f\|\right)^{d-1}\right) < \infty$, then*

$$\frac{1}{n_1 \ldots n_d} \sum_{i_1=0}^{n_1-1} \cdots \sum_{i_d=0}^{n_d-1} f(T_1^{i_1} \ldots T_d^{i_d}) \to \mathbb{E}(f|\mathcal{I}) \text{ a.s. as } \mathbf{n}_{\min} \to \infty$$

where $\mathcal{I} = \bigcap_{i=1}^{d} \mathcal{I}_{T_i}$. *Moreover, if* T_s *is ergodic for some* s *belonging to* $\{1, 2, \ldots, d\}$, *then* $\mathbb{E}(f|\mathcal{I}) = \mathbb{E}f$ *a.s.*

2.3. The multidimensional Birkhoff's ergodic theorem for random sets

The following is the "lim inf" part of Mosco convergence of two-dimensional Birkhoff's ergodic theorem for random sets.

2.3.3 Proposition. *Let* F *be a random set satisfying* $\mathbb{E}(\|F\| \log^+ \|F\|) < \infty$ *and let* T_1, T_2 *be two commutative measure-preserving transformations such that for every* $i \in \{1, 2\}$ *and every* $s \geq 1$, T_i^s *is ergodic. Then,*

$$\overline{\text{co}}\mathbb{E}F \subset s\text{-}\liminf_{m \wedge n \to \infty} \frac{1}{mn} \text{cl} \sum_{i=1}^{m} \sum_{j=1}^{n} F(T_1^i T_2^j(\omega)) \text{ a.s.}$$

Without assuming ergodicity on the measure-preserving transformations, we have the following theorem.

2.3.4 Theorem. *Let* $T_1, T_2, \ldots, T_d$ *be the commutative measure-preserving transformations. If* F *is the random set satisfying* $\mathbb{E}\left(\|F\| \left(\log^+ \|F\|\right)^{d-1}\right) < \infty$, *then*

$$\mathbb{E}(F|\mathcal{I}) \subset s\text{-}\liminf_{\mathbf{n}_{\min} \to \infty} \frac{1}{n_1 \ldots n_d} \text{cl} \sum_{i_1=0}^{n_1-1} \cdots \sum_{i_d=0}^{n_d-1} F(T_1^{i_1} \ldots T_d^{i_d}(\omega)) \text{ a.s.,}$$

where $\mathcal{I} = \bigcap_{i=1}^{d} \mathcal{I}_{T_i}$.

The following proposition is the "lim sup" part of Mosco convergence of two-dimensional multivalued Birkhoff's ergodic theorem.

2.3.5 Proposition. *Let* F *be a random set satisfying* $\mathbb{E}(\|F\| \log^+ \|F\|) < \infty$ *and let* T_1, T_2 *be two commutative measure-preserving transformations such that* T_i *is ergodic for some* i *belonging to* $\{1, 2\}$. *Then,*

$$w\text{-}\limsup_{m \wedge n \to \infty} \frac{1}{mn} \text{cl} \sum_{i=1}^{m} \sum_{j=1}^{n} F(T_1^i T_2^j(\omega)) \subset \overline{\text{co}}\mathbb{E}F \text{ a.s.}$$

The following is two-dimensional multivalued Birkhoff's ergodic theorem with respect to Mosco convergence and Wijsman convergence.

2.3.6 Theorem. *Let* F *be a random set satisfying* $\mathbb{E}\left(\|F\| \log^+ \|F\|\right) < \infty$ *and let* T_1, T_2 *two commutative measure-preserving transformations such that* T_i^s *is ergodic for every* $i \in \{1, 2\}$ *and every* $s \geq 1$. *Then*

$$\frac{1}{mn} \text{cl} \sum_{i=1}^{m} \sum_{j=1}^{n} F\left(T_1^i T_2^j(\omega)\right) \to \overline{\text{co}}\mathbb{E}F \text{ a.s. as } m \wedge n \to \infty$$

with respect to the topologies Mosco and Wijsman.

2.4. The two-dimensional Birkhoff's ergodic theorem for fuzzy random sets

In this section, by using Theorem 2.3.6, we obtain the Birkhoff's ergodic theorem for fuzzy random sets with respect to Mosco convergence.

2.4.1 Theorem. *Let T_1, T_2 be two commutative measure-preserving transformations such that T_i^s is ergodic for every $i \in \{1,2\}$ and for every positive integer s. If $\tilde{F} : \Omega \to \mathbf{F}(\mathfrak{X})$ is a fuzzy random set satisfying $\mathcal{S}^1_{L_1\tilde{F}} \neq \emptyset$, $\mathbb{E}\left(\|\mathrm{cl}(L_{0^+}\tilde{F})\| \log^+ \|\mathrm{cl}(L_{0^+}\tilde{F})\|\right) < \infty$ and*

$$L_\alpha(\overline{\mathrm{co}}\mathbb{E}\tilde{F}) = \mathrm{cl}(L_{\alpha^+}(\overline{\mathrm{co}}\mathbb{E}\tilde{F})) \text{ for all } \alpha \in [0,1] \setminus \mathbb{Q}, \tag{2.4.1}$$

then

$$\text{M-}\lim_{m\wedge n\to\infty} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{F}\left(T_1^i T_2^j(\omega)\right) = \overline{\mathrm{co}}\mathbb{E}\tilde{F} \text{ a.s.},$$

that is, there exists an $N \in \mathcal{A}$ with probability 0 such that

$$\text{M-}\lim_{m\wedge n\to\infty} L_\alpha\left(\frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{F}\left(T_1^i T_2^j(\omega)\right)\right) = L_\alpha\left(\overline{\mathrm{co}}\mathbb{E}\tilde{F}\right)$$

for every $\alpha \in (0,1]$ and every $\omega \in \Omega \setminus N$.

The following two examples show that all conditions of Theorem 2.4.1 are satisfied.

2.4.2 Example. Let $\mathfrak{X} = \mathbb{R}$ and $a < b$ $(a, b \in \mathbb{R})$. Assume that $u : \mathbb{R} \to [0,1]$ the fuzzy set in $\mathbb{R}$ satisfying u is an increasing function in $[a,b]$, $u(x) = 0$ for every $x \in (-\infty, a) \cup (b, +\infty)$ and $u(b) = 1$. For example,

$$u(x) = \begin{cases} \frac{x-a}{b-a} & \text{if } x \in [a,b], \\ 0 & \text{if } x \in (-\infty, a) \cup (b, +\infty). \end{cases}$$

The fuzzy random set $\tilde{F} : \Omega \to \mathbf{F}(\mathbb{R})$ is defined by $\tilde{F}(\omega) = u$ for every $\omega \in \Omega$. Then, $\tilde{F}$ satisfies all conditions of Theorem 2.4.1.

2.4.3 Example. Let $\mathfrak{X} = \mathbb{R}$. The fuzzy set $u : \mathbb{R} \to [0,1]$ is defined by

$$u(x) = \begin{cases} 0 & \text{if } x < 0, \\ 2x & \text{if } 0 \le x \le \frac{1}{2}, \\ 2(1-x) & \text{if } \frac{1}{2} < x < 1, \\ 0 & \text{if } x \ge 1. \end{cases}$$

Then, $\tilde{F}$ satisfies all conditions of Theorem 2.4.1, where the fuzzy random set $\tilde{F} : \Omega \to \mathbf{F}(\mathbb{R})$ is defined by $\tilde{F}(\omega) = u$ for every $\omega \in \Omega$.

The following example will show that in Theorem 2.4.1, the condition (2.4.1) does not follow from the other conditions.

2.4.4 Example. Let $\mathfrak{X} = \mathbb{R}$. We define the fuzzy set $u : \mathbb{R} \to [0,1]$ as follows:

$$u(x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ 2\sqrt{2}x & \text{if} \quad 0 \le x \le \frac{1}{4}, \\ \frac{\sqrt{2}}{2} & \text{if} \quad \frac{1}{4} < x < \frac{3}{4}, \\ (4 - 2\sqrt{2})x - 3 + 2\sqrt{2} & \text{if} \quad \frac{3}{4} \le x \le 1, \\ 0 & \text{if} \quad x > 1. \end{cases}$$

Next, the fuzzy random set $\tilde{F}$ is defined by $\tilde{F}(\omega) = u$ for every $\omega \in \Omega$.

It is not hard to check that $L_\alpha u \neq \mathrm{cl}(L_{\alpha^+} u)$ for $\alpha = \frac{\sqrt{2}}{2}$. Therefore, the condition (2.4.1) in Theorem 2.4.1 is not satisfied. We can show that the other conditions are satisfied. Hence, the condition (2.4.1) does not follow from the other conditions.

The conclusions of Chapter 2

In this chapter, we obtain some main results:

- Establish the multidimensional Birkhoff's ergodic theorem for separable real Banach space-valued random variables.

- Establish the two-dimensional Birkhoff's ergodic theorem for random sets and for fuzzy random sets.

- Establish the multidimensional multi-valued Birkhoff's ergodic theorem without assuming ergodicity on the measure-preserving transformations.

- Give some illustrative examples.

CHAPTER 3

THE STRONG LAWS OF LARGE NUMBERS FOR DOUBLE ARRAY OF RANDOM SETS

In this chapter, we establish some strong laws of large numbers for double array of random sets with respect to the topologies Mosco and Wijsman. Main results of this chapter are based on the papers [1] and [2].

3.1. Preliminaries

In the following, we give a key lemma to establish the strong law of large numbers for double array of random sets with respect to the convergence as $m \vee n \to \infty$.

3.1.4 Lemma. *Let $\{x_{ij} : i \geq 1, j \geq 1\}$ be a double array of elements in a Banach space such that*

$$(i) \text{ for each } m \geq 1, \quad \frac{1}{n}\sum_{j=1}^{n} x_{mj} \to x \quad \text{as } n \to \infty,$$

$$(ii) \text{ for each } n \geq 1, \quad \frac{1}{m}\sum_{i=1}^{m} x_{in} \to x \quad \text{as } m \to \infty,$$

$$(iii) \ \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} \to x \quad \text{as } m \wedge n \to \infty,$$

then

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} \to x \quad \text{as } m \vee n \to \infty.$$

By Lemma 3.1.4, we prove the two-indexed type of Stolz's theorem.

3.1.5 Lemma. *Let $\{x_{ij} : i \geq 1, j \geq 1\}$ be a double array of elements in Banach space. If $\lim_{i \vee j \to \infty} x_{ij} = x$ then*

$$\lim_{m \vee n \to \infty} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = x.$$

3.2. The strong law of large numbers for double array of random sets

The following theorem is an extension of C. Hess's results (in 1985 and in 1999) and F. Hiai's result (in 1985) from the sequence case to the double array case.

3.2.1 Theorem. *If* $\{F_{ij} : i \geq 1, j \geq 1\}$ *is the double array of p.i.i.d such that* $\mathcal{S}^1_{F_{11}} \neq \emptyset$ *and* $\mathbb{E}(\|F_{11}\| \log^+ \|F_{11}\|) < \infty$, *then*

$$\frac{1}{mn}\mathrm{cl}\sum_{i=1}^{m}\sum_{j=1}^{n} F_{ij}(\omega) \to \overline{\mathrm{co}}\mathbb{E}F_{11} \quad a.s. \ \ as \ \ m \vee n \to \infty$$

with respect to Mosco convergence and Wijsman convergence.

The following theorem is the strong law of large numbers for double array of independent random sets taking closed-values of Rademacher type p Banach space. In the sequence case, it is proved by F. Hiai in 1985.

3.2.2 Theorem. *Suppose that* $\mathfrak{X}$ *is a Rademacher type* p *Banach space* ($p \in [1, 2]$). *If* $\{F_{ij} : i \geq 1, j \geq 1\}$ *is a double array of independent random sets satisfying*

$$(a) \qquad \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{\mathbb{E}\|F_{ij}\|^p}{(ij)^p} < \infty,$$

(b) *there exists an* $X \in c(\mathfrak{X})$ *such that*

$$X \subset s\text{-}\liminf_{i\vee j\to\infty}(\mathrm{cl}(\mathbb{E}(F_{ij}, \mathcal{A}_{F_{ij}}))),$$
$$\limsup_{i\vee j\to\infty} s(x^*, \mathrm{cl}(\mathbb{E}F_{ij})) \leq s(x^*, X), \text{ for every } x^* \in \mathfrak{X}^*,$$

then we obtain the strong law of large numbers with respect to Mosco convergence and Wijsman convergence

$$\frac{1}{mn}\mathrm{cl}\sum_{i=1}^{m}\sum_{j=1}^{n} F_{ij}(\omega) \to \overline{\mathrm{co}}X \quad a.s. \ \ as \ \ m \vee n \to \infty.$$

To establish the strong law of large numbers for double array 2-exchangeable random sets, we prove some following results of multi-indexed array of 2-exchangeable random elements.

3.2.6 Theorem. *Suppose that* $\{f_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\}$ *is an array of* 2*-exchangeable random elements taking values in* $\mathfrak{X}$. *If* $\mathbb{E}(\|f_{\mathbf{1}}\|(\log^+ \|f_{\mathbf{1}}\|)^{d-1}) < \infty$ *then*

$$\frac{1}{|\mathbf{n}|}\sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{n}} f_{\mathbf{i}} \to f \quad a.s. \ and \ in \ \mathbf{L}^1 \quad as \ \ \mathbf{n}_{\max} \to \infty,$$

for some random element f *satisfying* $\mathbb{E}f = \mathbb{E}f_{\mathbf{1}}$.

Next, we obtain the following result that the limit of the average will be a constant.

3.2.7 Theorem. *Assume that* $\{f_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\}$ *is an array of* 2*-exchangeable random elements in* $\mathbf{L}^2(\mathfrak{X})$ *and* $\mathfrak{X}^*$ *is separable. If* $\mathrm{Cov}(\langle x^*, f_{\mathbf{1}}\rangle, \langle x^*, f_{\mathbf{2}}\rangle) = 0$ *for every* $x^* \in \mathfrak{X}^*$*, then*

$$\frac{1}{|\mathbf{n}|}\sum_{\mathbf{i=1}}^{\mathbf{n}} f_{\mathbf{i}} \to \mathbb{E}f_{\mathbf{1}} \ \ a.s. \ \ and \ in \ \mathbf{L}^1 \ \ as \ \mathbf{n}_{\max} \to \infty.$$

The following result is an extension of Theorem 3.2.6 to double array of random sets.

3.2.8 Theorem. *Assume that* $\{F_{ij} : i \geq 1, j \geq 1\}$ *is a double array of* 2*-exchangeable random sets such that* $\mathcal{S}^1_{F_{11}} \neq \emptyset$ *and* $\mathbb{E}(\|F_{11}\| \log^+ \|F_{11}\|) < \infty$*. Let* $S_{mn} = \sum_{i=1}^m \sum_{j=1}^n F_{ij}$*. Then,*

(a) $\overline{\mathrm{co}}\mathbb{E}F_{11} \subset \mathrm{cl}(\mathbb{E}F)$*, where* F *is the random set satisfying* $F(\omega) = s\text{-}\liminf_{m\wedge n\to\infty} \mathrm{cl}\left(\frac{S_{mn}(\omega)}{mn}\right)$ *a.s.*

(b) if $\mathfrak{X}$ *is reflexive and* $\sup_{m,n\geq 1} \|F_{mn}(\omega)\| < \infty$ *a.s. then* $\mathrm{cl}(\mathbb{E}Y) \subset \overline{\mathrm{co}}\mathbb{E}F_{11}$*, where* Y *is the random set satisfying* $Y(\omega) = w\text{-}\limsup_{m\vee n\to\infty} \mathrm{cl}\left(\frac{S_{mn}(\omega)}{mn}\right)$ *a.s.*

The conclusions of Chapter 3

In this chapter, we obtain some main results:

- Give a condition of the convergence for double array of elements in a Banach space as $m \vee n \to \infty$, based on the convergence for each row and for each column and as $m \wedge n \to \infty$.

- Establish the strong law of large numbers for multi-indexed array of 2-exchangeable random variables taking values in separable real Banach space.

- Establish the strong laws of large numbers on Mosco convergence and Wijsman convergence for double array of random sets with the cases: p.i.i.d, or independent and taking closed-values of Rademacher type p Banach space, or 2-exchangeable.

- Give an illustrative example.

CHAPTER 4

THE STRONG LAW OF LARGE NUMBERS FOR TRIANGULAR ARRAY OF RANDOM SETS

In this chapter, we establish some strong law of large numbers for triangular array of rowwise independent random sets taking closed-values of Rademacher type p Banach space. We consider the types of convergence: Mosco and Wijsman. Main results of this chapter are based on the paper [4].

4.1. Stolz's theorem for the triangular array case

For $\{x_{ni} : n \geq 1, 1 \leq i \leq n\} \subset \mathbb{R}$, we put

$$\liminf_{i\to\infty} x_{ni} = \sup_{k\geq 1} \inf_{k\leq i\leq n} x_{ni},$$
$$\limsup_{i\to\infty} x_{ni} = \inf_{k\geq 1} \sup_{k\leq i\leq n} x_{ni}.$$

4.1.1 Definition. (a) A triangular array $\{x_{ni} : n \geq 1, 1 \leq i \leq n\} \subset \mathbb{R}$ *converges to* $x \in \mathbb{R}$ *as* $i \to \infty$, denoted by $\lim\limits_{i\to\infty} x_{ni} = x$, if

$$\liminf_{i\to\infty} x_{ni} = \limsup_{i\to\infty} x_{ni} = x.$$

(b) A triangular array $\{x_{ni} : n \geq 1, 1 \leq i \leq n\} \subset \mathfrak{X}$ *converges to* $x \in \mathfrak{X}$ *as* $i \to \infty$, denoted by $\lim\limits_{i\to\infty} x_{ni} = x$, if $\lim\limits_{i\to\infty} \|x_{ni} - x\| = 0$.

The following lemma is Stolz's theorem for the triangular array case.

4.1.3 Lemma. *Let* $\{x_{ni} : n \geq 1, 1 \leq i \leq n\}$ *be a triangular array of elements in a Banach space satisfying*

(a) $\lim\limits_{i\to\infty} x_{ni} = x$,

(b) *there exists a constant* $C > 0$ *such that* $\|x_{ni}\| \leq C$, *for every* $n \geq 1, 1 \leq i \leq n$.

Then

$$\frac{1}{n}\sum_{i=1}^{n} x_{ni} \to x \ \ \text{as}\, n \to \infty.$$

In this section, we also give an example that Lemma 4.1.3 without condition (b) is not true.

4.2. The strong law of large numbers for triangular array of random sets

The family of random sets $\{F_i : i \in I\}$ is called *bounded expectation* if there exists a positive constant C such that $\|\mathbb{E}F_i\| \leq C$ for every $i \in I$.

The following theorem is as similarly as F. Hiai's result (in 1985) for triangular array case.

4.2.1 Theorem. *Suppose that $\mathfrak{X}$ is a Rademacher type p Banach space ($p \in (1,2]$) and $\{F_{ni} : n \geq 1, 1 \leq i \leq n\}$ is a triangular array of rowwise independent, bounded expectation random sets. Let $\Psi(t) : \mathbb{R} \to \mathbb{R}$ be a positive, even, convex, continuous function such that*

$$\frac{\Psi(|t|)}{|t|^r} \uparrow \text{ and } \frac{\Psi(|t|)}{|t|^{r+p-1}} \downarrow \text{ as } |t| \uparrow \tag{4.2.1}$$

for some nonnegative integer r and there exists a positive constant C_1 satisfying

$$\Psi(a+b) \leq C_1(\Psi(a) + \Psi(b)) \text{ for every } a, b \in \mathbb{R}. \tag{4.2.2}$$

If the following conditions are satisfied

$$+) \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{\mathbb{E}(\Psi(\|F_{ni}\|))}{\Psi(n)} < \infty, \tag{4.2.3}$$

$$+) \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} \frac{\mathbb{E}\|F_{ni}\|^p}{n^p} \right)^{p.k} < \infty, \tag{4.2.4}$$

for some positive integer k and there exists an $X \in c(\mathfrak{X})$ such that

$$+) \; X \subset s\text{-}\liminf_{i\to\infty} \mathrm{cl}(\mathbb{E}(F_{ni}, \mathcal{A}_{F_{ni}})),$$
$$+) \; \limsup_{i\to\infty} s(x^*, \mathrm{cl}(\mathbb{E}F_{ni})) \leq s(x^*, X), \text{ for every } x^* \in \mathfrak{X}^*$$

then we obtain the strong law of large numbers

$$\frac{1}{n}\mathrm{cl}\sum_{i=1}^{n} F_{ni}(\omega) \to \overline{\mathrm{co}}X \text{ a.s. as } n \to \infty$$

with respect to Mosco convergence and Wijsman convergence.

4.2.2 Remark. In Theorem 4.2.1, if the condition (4.2.1) is satisfied for $r = 0$ or $r = 1$ then we can remove the condition (4.2.4).

We also give an example that the condition "bounded expectation" in Theorem 4.2.1 does not follow from the other conditions.

The following theorem is an extension of the result of A. Bozorgnia, R. F. Patterson and R. L. Taylor (in 1997) for random sets.

4.2.3 Theorem. *Assume that* $\{F_{ni} : n \geq 1, 1 \leq i \leq n\}$ *is a triangular array of rowwise independent random sets taking closed-values of Rademacher type* p *Banach space* $(p \in (1, 2])$. *Let* $\{a_n : n \geq 1\}$ *be a increasing sequence of positive real numbers such that* $\lim_{n\to\infty} a_n = +\infty$ *and let* $\Psi(t)$ *be a positive, even, continuous function such that*

$$\frac{\Psi(|t|)}{|t|^r} \uparrow \text{ and } \frac{\Psi(|t|)}{|t|^{r+p-1}} \downarrow \text{ as } |t| \uparrow \tag{4.2.23}$$

for some nonnegative integer r. *If*

$$+)\ 0 \in \mathbb{E}(F_{ni}, \mathcal{A}_{F_{ni}}), \tag{4.2.24}$$

$$+)\ \sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{\mathbb{E}(\Psi(\|F_{ni}\|))}{\Psi(a_n)} < \infty, \tag{4.2.25}$$

$$+)\ \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} \frac{\mathbb{E}\|F_{ni}\|^p}{a_n^p}\right)^{p.k} < \infty, \tag{4.2.26}$$

for some positive integer k, *then*

$$0 \in s\text{-}\liminf_{n\to\infty} \frac{1}{a_n}\mathrm{cl}\sum_{i=1}^{n} F_{ni}(\omega) \text{ a.s.}$$

4.2.4 Remark. In Theorem 4.2.3, if the condition (4.2.23) is satisfied for $r = 1$ then we can remove the condition (4.2.26), and if the condition (4.2.23) is satisfied for $r = 0$ then we can remove the conditions (4.2.24), (4.2.26).

We also give an example that the conclusion of Theorem 4.2.3 cannot be replaced by by the stronger one

$$\text{M-}\lim \frac{1}{a_n}\mathrm{cl}\sum_{i=1}^{n} F_{ni}(\omega) = \{0\} \text{ a.s.}$$

The conclusions of Chapter 4

In this chapter, we obtain some main results:

- Establish Stolz's theorem for the triangular array case.

- Establish the strong law of large numbers for triangular array of rowwise independent random sets taking closed-values of Rademacher type p Banach space with respect to Mosco convergence and Wijsman convergence.

- Give some illustrative examples.

GENERAL CONCLUSIONS AND SUGGESTIONS

1. General conclusions

In this thesis, we obtain some main results:

- Establish some results of convergence for multi-indexed array of closed subsets of a Banach space, and for multi-indices array of random sets.

- Establish the multidimensional Birkhoff's ergodic theorem for separable real Banach space-valued random variables.

- Establish the two-dimensional multi-valued Birkhoff's ergodic theorem with respect to Mosco convergence and Wijsman convergence.

- Establish the strong law of large numbers for multi-indexed array of 2-exchangeable random elements.

- Establish the strong laws of large numbers on Mosco convergence and Wijsman convergence for double array of random sets with the cases: p.i.i.d, or independent and taking closed-values of Rademacher type p Banach space, or 2-exchangeable.

- Establish the strong law of large numbers for triangular array of rowwise independent random sets taking closed-values of Rademacher type p Banach space with respect to Mosco convergence and Wijsman convergence.

2. Recommendations

In the near future we will study the following issues:

- The weak law of large numbers on Mosco convergence and Wijsman convergence for sequence and for array of random sets.

- The multi-valued ergodic theorems with respect to the topologies: Mosco, Wijsman, Slice, Hausdorff, etc. for the one-dimensional case and for the multi-dimensional case.

LIST OF THE AUTHOR'S ARTICLES RELATED TO THE THESIS

1. C. Castaing, N. V. Quang and D. X. Giap (2012), Mosco convergence of strong law of large numbers for double array of closed valued random variables in Banach space, *Journal of Nonlinear and Convex Analysis*, **13**(4), 615-636.

2. C. Castaing, N. V. Quang and D. X. Giap (2012), Various convergence results in strong law of large numbers for double array of random sets in Banach spaces, *Journal of Nonlinear and Convex Analysis*, **13**(1), 1-30.

3. D. X. Giap and N. V. Quang (2015), Multidimensional and multivalued ergodic theorems for measure-preserving transformations, *Set-Valued and Variational Analysis*, DOI 10.1007/s11228-016-0361-z (*Available online January 2016*).

4. N. V. Quang and D. X. Giap (2013), Mosco convergence of SLLN for triangular arrays of rowwise independent random sets, *Statistics and Probability Letters*, **83**, 1117-1126.

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- VMS - SMF Joint Congress (Hue University of Education, 20-24/08/2012),

- The 8th Vietnamese Mathematical Conference (University of Information Officers, 10-14/08/2013),

- The 5th National Conference: "Probability - Statistics: Research, Application and Teaching" (Da Nang University of Education, 23-25/05/2015),

- Seminar of Department of Probability, Statistics and Application, Faculty of Mathematics, Vinh University (from 2011 to 2015).